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LOCAL RECOGNITION OF NON-INCIDENT POINT-HYPERPLANE GRAPHS

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Let $\mathbb{P}$ be a projective space. By $\mathbf{H}(\mathbb{P})$ we denote the graph whose vertices are the non-incident point-hyperplane pairs of $\mathbb{P}$, two vertices (p,H) and (q,I) being adjacent if and only if $p \in I$ and $q \in H$. In this paper we give a characterization of the graph $\mathbf{H}(\mathbb{P})$ (as well as of some related graphs) by its local structure. We apply this result by two characterizations of groups G with $\mathrm{PSL}_n(\mathbb{F}) \leq G \leq \mathrm{PGL}_n(\mathbb{F})$, by properties of centralizers of some (generalized) reflections. Here $\mathbb{F}$ is the (skew) field of coordinates of $\mathbb{P}$.

1. Introduction

Local recognition of graphs is a problem described, for example, in [1]. The general idea is the following. Choose your favorite graph Δ and try to find all connected graphs Γ that are *locally* Δ, i.e., graphs whose induced subgraph on the set of all neighbors of an arbitrary vertex of Γ is isomorphic to Δ. One restricts the search to connected graphs, because a graph is locally Δ if and only if all of its connected components are locally Δ. There has already been done a lot of work in this direction; besides the papers referred to in [1], we recommend [7,8].

Suppose $\mathbb{P}$ is a projective space of (projective) dimension n (possibly infinite). Then by $\mathbf{H}(\mathbb{P})$ we denote the graph with as vertices the non-incident point-hyperplane pairs and with two vertices (p,H) and (q,I), with p,q points and H,I hyperplanes such that $p \notin H$ and $q \notin I$, being adjacent if and only if $p \in I$ and $q \in H$.

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For each vertex of the graph $\mathbf{H}(\mathbb{P})$, the induced subgraph on the neighbors of this vertex is isomorphic to $\mathbf{H}(\mathbb{P}_0)$, where $\mathbb{P}_0$ is a hyperplane of $\mathbb{P}$. In this paper we give a characterization of the graphs $\mathbf{H}(\mathbb{P})$ by their local structure.

In fact, we consider a slightly larger class of graphs. Let $\mathbb{H}$ be a subspace of the dual $\mathbb{P}^{\mathrm{dual}}$ of $\mathbb{P}$ with the property that the intersection of all the hyperplanes $H \in \mathbb{H}$ is trivial (we say $\mathbb{H}$ has a trivial annihilator in $\mathbb{P}$). If $\mathbb{P}$ is finite-dimensional, then $\mathbb{H}$ equals $\mathbb{P}^{\mathrm{dual}}$, but for infinite dimensional $\mathbb{P}$ the space $\mathbb{H}$ can be a proper subspace of $\mathbb{P}^{\mathrm{dual}}$. The subgraph $\mathbf{H}(\mathbb{P},\mathbb{H})$ of $\mathbf{H}(\mathbb{P})$ induced on the vertices (p,H) with $H \in \mathbb{H}$, has the property that for each vertex v the induced subgraph on the neighbors of v is isomorphic to $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$ for a suitable hyperplane $\mathbb{P}_0$ of $\mathbb{P}$ and a suitable subspace $\mathbb{H}_0$ of ${\mathbb{P}_0}^{\mathrm{dual}}$. Indeed, if $v=(x,X)$, then with $\mathbb{P}_0$ the projective space induced on X and $\mathbb{H}_0$ the set of hyperplanes K of $\mathbb{P}_0$ such that the subspace of $\mathbb{P}$ generated by x and K belongs to $\mathbb{H}$, we find the induced subgraph on the neighbors of v to be isomorphic to $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$. Moreover, as $\bigcap_{H\in\mathbb{H},x\in H} H=\{x\}$, we have $\bigcap_{H\in\mathbb{H}_0} H=\emptyset$. Our main result reads as follows.

Theorem 1.1. *Let $\mathbb{P}_0$ be a projective space of dimension at least 3 and $\mathbb{H}_0$ a subspace of ${\mathbb{P}_0}^{\mathrm{dual}}$ with trivial annihilator in $\mathbb{P}_0$. Suppose Γ is a connected graph which is locally $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$. Then Γ is isomorphic to $\mathbf{H}(\mathbb{P},\mathbb{H})$ for some projective space $\mathbb{P}$ and some subspace $\mathbb{H}$ of $\mathbb{P}^{\mathrm{dual}}$ with trivial annihilator in $\mathbb{P}$.*

The condition $\dim(\mathbb{P}_0)\geq 3$ in our local recognition result is sharp as is shown by an example in Section 5 of a connected, locally $\mathbf{H}(\mathbb{P}(\mathbb{F}_2^3))$ graph that is not isomorphic to $\mathbf{H}(\mathbb{P}(\mathbb{F}_2^4))$. Examples of graphs which are locally $\mathbf{H}(\mathbb{P}(\mathbb{F}_{2^m}^3))$, with $m>1$, but not isomorphic to $\mathbf{H}(\mathbb{P}(\mathbb{F}_{2^m}^4))$ have been constructed by Postma and Cohen, see [3].

Natural examples of locally recognizable graphs $\mathbf{H}(\mathbb{P},\mathbb{H})$ arise from functional analysis. Let V be an infinite-dimensional normed vector space over a field $\mathbb{F}$ and let $B(V,\mathbb{F})$ be the subspace of the dual space V^* (in the sense of linear algebra) of V consisting of the bounded linear functionals from V to $\mathbb{F}$. Then $\mathbf{H}(\mathbb{P}(V),\mathbb{P}(B(V,\mathbb{F})))$ is locally recognizable by Theorem 1.1.

Our proof of Theorem 1.1 is partly motivated by the methods developed in [4], where local recognition results are obtained for subgraphs of $\mathbf{H}(\mathbb{P})$ fixed under polarities of $\mathbb{P}$.

If $\mathbb{P}$ is the projective space $\mathbb{P}(V)$ of some vector space V defined over a field $\mathbb{F}$ of order at least 3, then the graphs $\mathbf{H}(\mathbb{P},\mathbb{H})$ can be described as graphs on the reflection tori in subgroups of $\mathrm{GL}(V)$. Let V be a left vector space over a (possibly commutative) skew field $\mathbb{F}$. For $g\in\mathrm{GL}(V)$, we set

$$[V,g]=\{vg-v \mid v\in V\} \quad \text{and} \quad C_V(g)=\{v\in V \mid vg-v=0\},$$

and call these subspaces the *center* and *axis* of g. A transformation $g \in \mathrm{GL}(V)$ satisfying $\dim([V,g]) = 1$ is called a *reflection* if $[V,g] \not\subseteq C_V(g)$. Observe that $C_V(g)$ is a hyperplane if g is a reflection.

If we specify a hyperplane H and a one-dimensional subspace, that is, a *projective point*, p of V, then by $T_{(p,H)}$ we denote the subgroup of $\mathrm{GL}(V)$ generated by all $g \in \mathrm{GL}(V)$ with $p = [V,g]$ and $H = C_V(g)$. If $p \notin H$, the subgroup $T_{(p,H)}$ consists of the identity and all reflections with center p and axis H. The group is isomorphic with $\mathbb{F}^*$ and is called a *reflection torus*. All reflection tori in $\mathrm{GL}(V)$ generate the full finitary general group $\mathrm{FGL}(V)$ of V, i.e., the subgroup of $\mathrm{GL}(V)$ consisting of all elements $g \in \mathrm{GL}(V)$ with $[V,g]$ finite dimensional. Below we will describe more examples of groups generated by reflection tori, closely related to the graphs appearing in Theorem 1.1.

Let Φ be a subspace of V^*. By $R(V,\Phi)$ we denote the subgroup G of $\mathrm{GL}(V)$ generated by the reflections with center in V and axis in Φ. If $\Phi = V^*$, then G is equal to the full finitary general linear group $\mathrm{FGL}(V)$. If $\Phi \neq V^*$ but $\{v \in V \mid v\phi = 0 \text{ for all } \phi \in \Phi\} = 0$ (i.e., the annihilator of Φ in V is trivial), then $R(V,\Phi)$ still acts irreducibly on V, see [2].

If $T_{(p,H)}$ and $T_{(q,I)}$ are two distinct reflection tori in $\mathrm{GL}(V)$, then $T_{(p,H)}$ and $T_{(q,I)}$ commute elementwise if and only if $p \in I$ and $q \in H$. Hence, if G is one of the groups $R(V,\Phi)$, where the annihilator of Φ in V is trivial, then the graph with as vertex set the reflection tori in G, two tori being adjacent if and only if they commute, is isomorphic to the graph $\mathbf{H}(\mathbb{P}(V),\mathbb{P}(\Phi))$.

If $\mathbb{F}$ is a commutative field and $\mathcal{C}$ is a conjugacy class of reflections in $\mathrm{GL}_{n+1}(\mathbb{F})$, then each reflection torus of $\mathrm{GL}_{n+1}(\mathbb{F})$ meets $\mathcal{C}$ in a unique element. So, the commuting graph on $\mathcal{C}$, i.e., the graph with vertex set $\mathcal{C}$ and in which two distinct vertices are adjacent if and only if they commute, is isomorphic to $\mathbf{H}(\mathbb{P}(\mathbb{F}^{n+1}))$.

In view of these observations we can use Theorem 1.1 in order to locally recognize linear groups. We state two such results.

Theorem 1.2. *Let $n \geq 3$ be finite, and let $\mathbb{F}$ be a skew field of order ≥ 3. Let G be a group with subgroups J, K, X, Y and distinct elements $x \in J$, $y \in K$ such that*

(i) $C_G(x) = X \times K$ with $K \cong \mathrm{GL}_{n+1}(\mathbb{F})$;
(ii) $C_G(y) = Y \times J$ with $J \cong \mathrm{GL}_{n+1}(\mathbb{F})$;
(iii) there exists an element in $J \cap K$ that is a reflection of both J and K conjugate to x in J and y in K, respectively.

If $G = \langle J, K\rangle$, then (up to isomorphism) $\mathrm{PSL}_{n+2}(\mathbb{F}) \leq G/Z(G) \leq \mathrm{PGL}_{n+2}(\mathbb{F})$.

Our second application deals with groups defined over fields. Let n be finite and $\mathbb{F}$ a field. An element r of $\mathrm{SL}_{n+1}(\mathbb{F})$ is called a *generalized reflection* if, up to a scalar factor, r is a reflection in $\mathrm{GL}_{n+1}(\mathbb{F})$, i.e., if there exists a reflection in $rZ(\mathrm{GL}_{n+1}(\mathbb{F}))$. The axis and center of a generalized reflection are, by definition, its eigenspaces of dimension n and 1, respectively, in the natural module. They are the axis and center of the unique reflection in $rZ(\mathrm{GL}_{n+1}(\mathbb{F}))$. The group generated by all generalized reflections with a given axis and center is called a *generalized reflection torus* and is isomorphic to $\mathbb{F}^*/\langle\lambda\in\mathbb{F}^* \mid \lambda^{n+1}=1\rangle$.

With this notion we have the following result.

Theorem 1.3. *Let $n\geq 3$ be finite, and let $\mathbb{F}$ be a field. Let $p\geq 2$ be an integer. Let G be a group with subgroups J and K and distinct elements $x\in J$, $y\in K$ of order p such that*

(i) $C_G(x)$ contains a characteristic subgroup K with $K\cong\mathrm{SL}_{n+1}(\mathbb{F})$;
(ii) $C_G(y)$ contains a characteristic subgroup J with $J\cong\mathrm{SL}_{n+1}(\mathbb{F})$;
(iii) there exists an element z in $J\cap K$ conjugate to x in J and y in K, respectively; moreover, z is a generalized reflection of both K and J.

If $G=\langle J,K\rangle$, then $G/Z(G)\cong\mathrm{PSL}_{n+2}(\mathbb{F})$.

The latter theorem is the kind of result that is useful in the classification of finite simple groups in that a quasi-simple group is recognized from a component in the centralizer of an element about which some fusion information is given.

The remainder of this paper is organized as follows. In the next two sections we derive various properties of the graphs $\mathbf{H}(\mathbb{P},\mathbb{H})$. In particular, we show that both $\mathbb{P}$ and $\mathbb{H}$ can be recovered from the graph $\mathbf{H}(\mathbb{P},\mathbb{H})$. As a consequence we are able to determine the full automorphism group of $\mathbf{H}(\mathbb{P},\mathbb{H})$. Then in Section 4 we prove Theorem 1.1. As mentioned before, in Section 5 a family of graphs which are locally $\mathbf{H}(\mathbb{P}(\mathbb{F}_2^3))$ is discussed and finally in Section 6 the two group-theoretical applications, Theorem 1.2 and 1.3, of Theorem 1.1 are discussed.

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2. The point-hyperplane graph

Definition 2.1. Consider a projective space $\mathbb{P}$ and a subspace $\mathbb{H}$ of the dual $\mathbb{P}^{\mathrm{dual}}$ of $\mathbb{P}$ with $\bigcap_{H\in\mathbb{H}} H=\emptyset$. The *point-hyperplane graph* $\mathbf{H}(\mathbb{P},\mathbb{H})$ is the graph whose vertices are the non-incident point-hyperplane pairs of $\mathbb{P}$ with the hyperplanes in $\mathbb{H}$, in which a vertex (a,A) is adjacent to another vertex (b,B) (in symbols, $(a,A)\perp(b,B)$) if and only if $a\in B$ and $b\in A$.

By definition, we have $\mathbf{x}\not\perp\mathbf{x}$, so the perp $\mathbf{x}^\perp$ of $\mathbf{x}$ of all vertices of $\mathbf{H}(\mathbb{P},\mathbb{H})$ in $\perp$ relation to $\mathbf{x}$ is the set of vertices in $\mathbf{H}(\mathbb{P},\mathbb{H})$ at distance one from $\mathbf{x}$. Moreover, for a set X of vertices, we define the *perp of* X as $X^\perp:=\bigcap_{\mathbf{x}\in X}\mathbf{x}^\perp$ with the understanding that $\emptyset^\perp=\mathbf{H}(\mathbb{P},\mathbb{H})$. The *double perp of* X is $X^{\perp\perp}:=(X^\perp)^\perp$.

The graph $\mathbf{H}(\mathbb{P},\mathbb{P}^{\mathrm{dual}})$ is also denoted by $\mathbf{H}(\mathbb{P})$. Moreover, if $\mathbb{P}=\mathbb{P}(V)$ for some $(n+1)$-dimensional vector space V over a (skew) field $\mathbb{F}$, then we also write $\mathbf{H}_n(\mathbb{F})$ for $\mathbf{H}(\mathbb{P})$. If the field $\mathbb{F}$ is finite of order q, then we write $\mathbf{H}_n(q)$. Finally, if the field $\mathbb{F}$ is irrelevant, then we also write $\mathbf{H}_n$ instead of $\mathbf{H}_n(\mathbb{F})$.

Let $\mathbb{P}$ be a projective space and $\mathbb{H}$ a subspace of the dual of $\mathbb{P}$ such that the intersection over all hyperplanes in $\mathbb{H}$ is empty. A point p of the projective space $\mathbb{P}=(\mathcal{P},\mathcal{L})$ determines the set of vertices $v_p=\{(x,X)\in\mathbf{H}(\mathbb{P},\mathbb{H})\mid x=p\}$ of the graph $\mathbf{H}(\mathbb{P},\mathbb{H})$. A line l of $\mathbb{P}$ determines the union v_l of all sets v_p of vertices for $p\in l$. Clearly the map $v:\mathcal{P}\cup\mathcal{L}\to 2^{\mathbf{H}(\mathbb{P},\mathbb{H})}:x\mapsto v_x$ is injective, and $p\in l$ if and only if $v_p\subset v_l$, so we can identify the projective space with its image under v in the collection of all subsets of the vertex set of $\mathbf{H}(\mathbb{P},\mathbb{H})$. We shall refer to this image in $2^{\mathbf{H}(\mathbb{P},\mathbb{H})}$ as the *exterior projective space on* $\mathbf{H}(\mathbb{P},\mathbb{H})$. Similarly, one can map points Π and lines Λ of $\mathbb{H}$ onto subsets of vertices of $\mathbf{H}(\mathbb{P},\mathbb{H})$ of the form $w_\Pi=\{(x,X)\in\mathbf{H}(\mathbb{P},\mathbb{H})\mid X=\Pi\}$ and $w_\Lambda=\bigcup_{\Pi\supset\Lambda}w_\Pi$ for Π running over all points of $\mathbb{P}^{\mathrm{dual}}$ containing Λ. This gives rise to the *dual exterior projective space on* $\mathbf{H}(\mathbb{P},\mathbb{H})$. The subsets v_p, v_l, w_Λ and w_Π so obtained are called *exterior points*, *exterior lines*, *exterior hyperlines*, and *exterior hyperplanes* of $\mathbf{H}(\mathbb{P},\mathbb{H})$, respectively. Note that, if the projective space $\mathbb{P}$ is isomorphic to $\mathbb{H}$, there is an automorphism of $\mathbf{H}(\mathbb{P},\mathbb{H})$ mapping the image under v onto the image under w. (If π is an isomorphism from $\mathbb{P}$ to $\mathbb{H}$, then $(x,X)\mapsto(\pi(X),\pi(x))$ is an automorphism of $\mathbf{H}(\mathbb{P},\mathbb{H})$ as required.) Also, if $\mathbb{P}$ is a subspace of $\mathbb{H}^{\mathrm{dual}}$ with trivial annihilator in $\mathbb{H}$ (in particular, if $\mathbb{P}$ and $\mathbb{H}$ have the same finite dimension), then $\mathbf{H}(\mathbb{P},\mathbb{H})\cong\mathbf{H}(\mathbb{H},\mathbb{P})$ by the map $(x,X)\mapsto(X,x)$. So, in general it will not be possible to distinguish exterior points from exterior hyperplanes if one tries to reconstruct the projective space from the graph. Another useful observation is that the exterior points partition the vertex set of $\mathbf{H}(\mathbb{P},\mathbb{H})$. In other words, each vertex of $\mathbf{H}(\mathbb{P},\mathbb{H})$ belongs to a unique exterior point. The same holds for exterior hyperplanes.

One of our goals is to characterize the graph $\mathbf{H}(\mathbb{P},\mathbb{H})$ by its local structure. In this light the following two observations are important.

Proposition 2.2. *Let $\mathbb{P}$ have dimension at least one. The graph $\mathbf{H}(\mathbb{P},\mathbb{H})$ is locally $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$ for some hyperplanes $\mathbb{P}_0$ of $\mathbb{P}$ and $\mathbb{H}_0$ of $\mathbb{H}$.*

Proof. Let $\mathbf{x} = (x,X)$ be a vertex of $\mathbf{H}(\mathbb{P},\mathbb{H})$. Identify X with $\mathbb{P}_0$. For any vertex $\mathbf{y}=(y,Y)$ adjacent to $\mathbf{x}$, we have $x\in Y$, $y\in X\setminus Y$, and $X\cap Y$ a hyperplane in both X and Y, so $(y,X\cap Y)$ belongs to $\mathbf{H}(X)$. We can identify the space of all hyperplanes of the form $X\cap Y$ of X where $x\in Y\in\mathbb{H}$ with a hyperplane $\mathbb{H}_0$ of $\mathbb{H}$. Hence, $(y,X\cap Y)$ belongs to $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$.

Conversely, for any vertex of $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$, i.e., for any non-incident pair (z,Z) consisting of a point z and a hyperline Z of $\mathbb{P}$ with $z\in X$, $Z\subseteq X$, the pair $(z,\langle Z,x\rangle)$ is a vertex of $\mathbf{x}^\perp$. (Indeed, $z\notin\langle Z,x\rangle$, since $x\notin X$.)

Clearly, the maps $(y,Y)\mapsto(y,X\cap Y)$ and $(z,Z)\mapsto(z,\langle Z,x\rangle)$ are each other's inverses. Moreover, the maps preserve adjacency and the proposition follows. ∎

Proposition 2.3. *The graph $\mathbf{H}_1$ is the disjoint union of cliques of size two; the diameter of $\mathbf{H}_2$ equals three; the diameter of $\mathbf{H}(\mathbb{P},\mathbb{H})$, where $\dim(\mathbb{P})\geq 3$, equals two. In particular, $\mathbf{H}(\mathbb{P},\mathbb{H})$ is connected for $\dim(\mathbb{P})\geq 2$.*

Proof. The statement about $\mathbf{H}_1$ is obvious. Let $\mathbf{x}=(x,X)$, $\mathbf{y}=(y,Y)$ be two non-adjacent vertices of $\mathbf{H}_2$. The intersection $X\cap Y$ is a point or a line, and xy is a point or a line. The vertices $\mathbf{x}$ and $\mathbf{y}$ have a common neighbor, i.e., they are at distance two, if and only if $X\cap Y\not\subseteq xy$. If $X\cap Y\subseteq xy$, however, it is easily seen, that they are at distance three. Indeed, choose $a\in X\setminus\{y\}$ and $b\in Y\setminus\{x\}$ with $ay\not\ni b$ and $bx\not\ni a$. Then (x,X), (a,bx), (b,ay), (y,Y) establishes a path of length three.

Now let $\mathbf{x}=(x,X)$, $\mathbf{y}=(y,Y)$ be two non-adjacent vertices of $\mathbf{H}(\mathbb{P},\mathbb{H})$, where $\dim(\mathbb{P})\geq 3$. The intersection $X\cap Y$ contains a line. Since $x\notin X$ and $y\notin Y$, we find a point $z\in X\cap Y$ and a hyperplane $Z\supseteq xy$ with $z\notin Z$ and, thus, a vertex (z,Z) adjacent to both $\mathbf{x}$ and $\mathbf{y}$. ∎

Our first main result will be a reconstruction theorem of the projective space from graphs isomorphic to the point-hyperplane graph $\mathbf{H}=\mathbf{H}(\mathbb{P},\mathbb{H})$ without making use of the coordinates, see the next section. This goal will be achieved by the study of double perps of two vertices, i.e., subsets of $\mathbf{H}=\mathbf{H}(\mathbb{P},\mathbb{H})$ of the form $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$. By n we denote the dimension of $\mathbb{P}$.

Lemma 2.4. *Let $\mathbf{x} = (x,X)$, $\mathbf{y} = (y,Y)$ be distinct vertices of $\mathbf{H}$ with $\{\mathbf{x},\mathbf{y}\}^\perp \neq \emptyset$. Then the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ equals the set of vertices $\mathbf{z}=(z,Z)$ of $\mathbf{H}$ with $z\in xy$ and $Z\supseteq X\cap Y$.*

Proof. Distinct vertices with non-empty perp only exist for $n\geq 2$. The vertices of $\{\mathbf{x},\mathbf{y}\}^\perp$ are precisely the non-incident point-hyperplane pairs (p,H) with $p\in X\cap Y$ and $H\supset xy$. Let $\Big\{(p_i,H_i)\in\{\mathbf{x},\mathbf{y}\}^\perp \,|\, i\in I\Big\}$ be the set of all these vertices, indexed by some set I. Now $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=(\{\mathbf{x},\mathbf{y}\}^\perp)^\perp$ consists of precisely those vertices $(z,Z)\in\mathbf{H}$ with $z\in\bigcap_{i\in I}H_i$ and $Z\supset\langle(p_i)_{i\in I}\rangle$. But since $\{\mathbf{x},\mathbf{y}\}^\perp\neq\emptyset$, we have $\bigcap_{i\in I}H_i=xy$ and $\langle(p_i)_{i\in I}\rangle=X\cap Y$, thus proving the claim. ∎

In order to recover the projective spaces $\mathbb{P}$ and $\mathbb{H}$ from the information contained in a graph $\Gamma\overset{\phi}{\cong}\mathbf{H}$, we have to recognize vertices $\mathbf{x}$, $\mathbf{y}$ of Γ with $x=y$ or, dually, $X=Y$, if $\phi(\mathbf{x})=(x,X)$, $\phi(\mathbf{y})=(y,Y)$. Clearly, $x=y$ and $X=Y$ if and only if the vertices $\mathbf{x}$, $\mathbf{y}$ are equal. To recognize the other cases, we make use of the following definition and lemma.

Recall that the *(projective) codimension* of a subspace X of a projective space $\mathbb{P}$ is the number of elements in a maximal chain of proper inclusions of subspaces properly containing X and properly contained in $\mathbb{P}$. For example, the codimension of a hyperplane of $\mathbb{P}$ equals 0.

Definition 2.5. Let $n\geq 2$. Vertices $\mathbf{x}=(x,X)$, $\mathbf{y}=(y,Y)$ of $\mathbf{H}(\mathbb{P},\mathbb{H})$ are *in relative position* (i,j) if

$$i=\dim\langle x,y\rangle \text{ and } j=\operatorname{codim}(X\cap Y)$$

where dim denotes the projective dimension and codim the projective codimension. Note that $i,j\in\{0,1\}$.

Lemma 2.6. *Let $n\geq 2$, and let $\mathbf{x}$, $\mathbf{y}\in\mathbf{H}$. Then the following assertions hold.*

(i) The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0,0)$ if and only if they are equal.

(ii) The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0,1)$ or $(1,0)$ if and only if they are distinct and the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is minimal with respect to containment, i.e., it does not contain two vertices with a non-empty strictly smaller double perp.

(iii) The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1,1)$ if and only if they are distinct and the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is not minimal.

Proof. Statement (i) is obvious. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0,1)$. Then $\{\mathbf{x},\mathbf{y}\}^\perp\neq\emptyset$ (since $n\geq 2$), and we can apply Lemma 2.4. We obtain $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{(z,Z)\in\mathbf{H}\,|\,z=x=y, Z\supseteq X\cap Y\}$, whence any pair of distinct vertices contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is in relative position $(0,1)$ and gives rise to the same double perp. Symmetry handles the case $(1,0)$. If $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1,1)$ and $\{\mathbf{x},\mathbf{y}\}^{\perp}=\emptyset$, then $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\mathbf{H}$, which is clearly not minimal. So let us assume $\{\mathbf{x},\mathbf{y}\}^{\perp}\neq\emptyset$. Again by Lemma 2.4, we have $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{(z,Z)\in\mathbf{H}\mid z\in xy, Z\supseteq X\cap Y\}$. This double perp contains a vertex that is at relative position $(0,1)$ to $\mathbf{x}$, and we obtain a double perp strictly contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$. Statements (ii) and (iii) now follow from the fact that distinct vertices $\mathbf{x}=(x,X)$ and $\mathbf{y}=(y,Y)$ are in relative position $(0,1)$, $(1,0)$, or $(1,1)$. ∎

3. Reconstruction of the projective space

This section will concentrate on the reconstruction of the projective spaces $\mathbb{P}$ and $\mathbb{H}$ from a graph Γ isomorphic to $\mathbf{H}(\mathbb{P},\mathbb{H})$. Abusing notation to some extent, we will sometimes speak of relative positions on Γ, but *only if we have fixed a particular isomorphism* $\Gamma\cong\mathbf{H}(\mathbb{P},\mathbb{H})$. Throughout the whole section, let $n=\dim(\mathbb{P})\geq 2$. Furthermore, let $\mathbb{F}$ be a division ring and $\Gamma\cong\mathbf{H}=\mathbf{H}(\mathbb{P},\mathbb{H})$.

Definition 3.1. Let $\mathbf{x}$, $\mathbf{y}$ be vertices of Γ. Write $\mathbf{x}\approx\mathbf{y}$ to denote that $\mathbf{x}$, $\mathbf{y}$ are equal or the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is minimal with respect to inclusion (in the class of double perps $\{\mathbf{u},\mathbf{v}\}^{\perp\perp}$ for vertices $\mathbf{u}$, $\mathbf{v}$ with $\mathbf{u}\neq\mathbf{v}$).

For a fixed isomorphism $\Gamma\cong\mathbf{H}$ the relation $\approx$ coincides with the relation 'being equal or in relative position $(1,0)$ or $(0,1)$' by Lemma 2.6(ii). What remains is the problem of distinguishing the dual cases $(0,1)$ and $(1,0)$.

Lemma 3.2. *On the vertex set of Γ, there are unique equivalence relations $\approx^p$ and $\approx^h$ such that $\approx$ equals $\approx^p\cup\approx^h$ and $\approx^p\cap\approx^h$ is the identity relation. Moreover, for a fixed isomorphism $\Gamma\cong\mathbf{H}_n$, we either have*

- *$\approx^p$ is the relation 'being equal or in relative position $(0,1)$', and $\approx^h$ is the relation 'being equal or in relative position $(1,0)$', or*
- *$\approx^p$ is the relation 'being equal or in relative position $(1,0)$', and $\approx^h$ is the relation 'being equal or in relative position $(0,1)$'.*

In other words, for a fixed isomorphism $\Gamma\cong\mathbf{H}(\mathbb{P},\mathbb{H})$ and up to interchanging $\approx^p$ and $\approx^h$, we may assume that $\approx^p$ stands for being equal or in relative position $(0,1)$ and $\approx^h$ stands for being equal or in relative position $(1,0)$.

Proof. As we have noticed after Definition 3.1, vertices $\mathbf{x}$, $\mathbf{y}$ of Γ are in relation $\approx$ if and only if their images (x,X) and (y,Y) in $\mathbf{H}$ are equal or in relative positions $(0,1)$ or $(1,0)$. Let us consider equivalence relations that are subrelations of $\approx$. Obviously, the identity relation is an equivalence relation. Moreover, the relation 'equal or in relative position $(0,1)$' and the relation 'equal or in relative position $(1,0)$' are equivalence relations. Now let us assume we have vertices $\mathbf{x}=(x,X)$, $\mathbf{y}=(y,Y)$, $\mathbf{z}=(z,Z)$ of $\Gamma\cong\mathbf{H}$ such that $\mathbf{x}$, $\mathbf{y}$ are in relative position $(0,1)$ and $\mathbf{x}$, $\mathbf{z}$ are in relative position $(1,0)$. Then $y\neq z$ and $Y\neq Z$ and $\mathbf{y}$, $\mathbf{z}$ cannot be in relative position $(0,1)$ or $(1,0)$. Consequently, if we want to find two sub-equivalence relations $\approx^p$ and $\approx^h$ of $\approx$ whose union equals $\approx$, then either of $\approx^p$ and $\approx^h$ has to be a subrelation of the relation 'equal or in relative position $(0,1)$' or of the relation 'equal or in relative position $(1,0)$'. The lemma is proved. ∎

Convention 3.3. From now on, we will always assume that, as soon as we fix an isomorphism $\Gamma\cong\mathbf{H}$, the relation $\approx^p$ corresponds to 'equal or in relative position $(0,1)$'.

Definition 3.4. Let $\mathbf{x}$ be a vertex of Γ. With $\approx^p$ and $\approx^h$ as in Lemma 3.2, we shall write $[\mathbf{x}]^p$ to denote the equivalence class of $\approx^p$ containing $\mathbf{x}$, and similarly we shall write $[\mathbf{x}]^h$ to denote the equivalence class of $\approx^h$ containing $\mathbf{x}$. We shall refer to $[\mathbf{x}]^p$ as the *interior point* on $\mathbf{x}$ and to $[\mathbf{x}]^h$ as the *interior hyperplane* on $\mathbf{x}$.

Lemma 3.5. *For a fixed isomorphism $\Gamma\cong\mathbf{H}(\mathbb{P},\mathbb{H})$, an interior point of Γ is the image of an exterior point of $\mathbf{H}(\mathbb{P},\mathbb{H})$ under this isomorphism, and vice versa. The same correspondence exists between interior hyperplanes of Γ and exterior hyperplanes of $\mathbf{H}(\mathbb{P},\mathbb{H})$.*

Proof. This is direct from the above. ∎

Note that an exterior point and an exterior hyperplane of $\mathbf{H}(\mathbb{P},\mathbb{H})$ are disjoint if and only if the corresponding point and hyperplane of $\mathbb{P}_n(\mathbb{F})$ are incident. The above lemma motivates us to call a pair (p,H) of an interior point and an interior hyperplane of Γ *incident* if and only if $p\cap H=\emptyset$. This enables us to define interior lines.

Definition 3.6. Let p and q be distinct interior points of Γ. The *interior line* l of Γ spanned by p and q is the union of all interior points disjoint from every interior hyperplane disjoint from both p and q. In other words, the interior line pq consists of exactly those interior points which are incident with every interior hyperplane incident with both p and q.

Dually, one can define the *interior hyperline* spanned by distinct interior hyperplanes H and I as the union of all interior hyperplanes disjoint from every interior point disjoint from both H and I.

Lemma 3.7. *For a fixed isomorphism $\Gamma \cong \mathbf{H}(\mathbb{P},\mathbb{H})$, each interior line of Γ is the image of an exterior line of $\mathbf{H}(\mathbb{P},\mathbb{H})$ under this isomorphism, and vice versa. The analogue holds for interior hyperlines.*

Proof. The proof is straightforward. ∎

The geometry $(\mathcal{P},\mathcal{L},\subset)$ on Γ where $\mathcal{P}$ is the set of interior points of Γ and $\mathcal{L}$ is the set of interior lines of Γ is called the *interior projective space on Γ*. By Lemma 3.5 and Lemma 3.7, this interior projective space is isomorphic to the exterior projective space on $\mathbf{H}(\mathbb{P},\mathbb{H})$. Proceeding with $\approx^h$ as we did for $\approx^p$, the same holds for the dual of the interior projective space on Γ. We summarize the findings in the following proposition.

Proposition 3.8. *Let $n \geq 2$. Up to interchanging $\approx^p$ and $\approx^h$ every isomorphism $\Gamma \cong \mathbf{H}(\mathbb{P},\mathbb{H})$ induces an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{H}(\mathbb{P},\mathbb{H})$. The analogue holds for the dual interior projective space on Γ.* ∎

Corollary 3.9. *Let $n \geq 2$, and let Γ be isomorphic to $\mathbf{H}(\mathbb{P},\mathbb{H})$. Then the interior projective space on Γ is isomorphic to $\mathbb{P}$ or $\mathbb{H}$.* ∎

Corollary 3.10. *Let $n \geq 2$, and let Γ be isomorphic to $\mathbf{H}(\mathbb{P})$. If $\mathbb{P}$ and $\mathbb{P}^{\mathrm{dual}}$ are isomorphic, then the automorphism group of Γ is of the form $\mathrm{Aut}(\mathbb{P}).2$. Otherwise, it is isomorphic to $\mathrm{Aut}(\mathbb{P})$.*

Proof. Indeed, every automorphism of $\mathbb{P}$ induces an automorphism of Γ. Conversely, every automorphism of Γ that preserves the interior projective space gives rise to a unique automorphism of $\mathbb{P}$, by the theorem. Moreover, every automorphism of Γ either preserves the interior projective space or maps it onto the dual interior projective space, again by the theorem. Finally, an outer automorphism is induced on Γ by the map $(p,H) \mapsto (\delta(H),\delta(p))$ for a duality δ of the projective space, and the map $(p,H) \mapsto (\delta^2(p),\delta^2(H))$ preserves the interior projective space on Γ. ∎

Remark 3.11. Now might be an appropriate moment to address the problem of duality. Although, by Convention 3.3, as soon as we fix an isomorphism $\Gamma \cong \mathbf{H}$, we also choose the equivalence relation $\approx^p$ to correspond to the relation 'equal or in relative position $(0,1)$' of $\mathbf{H}$, there is a subtle problem – mainly of notation – coming with this: Suppose $\Gamma \cong \mathbf{H}_n(\mathbb{F})$ with $\mathbb{F} \not\cong \mathbb{F}^{\mathrm{opp}}$ and $n \in \mathbb{N}$. Then, by the convention, the interior projective space on Γ will always be isomorphic to $\mathbb{P}_n(\mathbb{F})$. If one wants the interior projective space to be isomorphic to $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$, then one will have to fix an isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F}^{\mathrm{opp}})$, although $\mathbf{H}_n(\mathbb{F}) \cong \mathbf{H}_n(\mathbb{F}^{\mathrm{opp}})$ by means of the map

$(p,H)\mapsto(H,p)$. The reason for this is that we have defined the graph $\mathbf{H}_n(\mathbb{F})$ as the point-hyperplane graph of the space $\mathbb{P}_n(\mathbb{F})$, which by Convention 3.3 determines the isomorphism class of the interior projective space on Γ.

The remainder of this section serves as a collection of results to be used later on. First comes a useful result on subspaces of the interior projective space of Γ.

Lemma 3.12. *Let U be a finite dimensional subspace of the interior projective space on Γ. For any projective basis of U there exists a clique of vertices in Γ such that the interior points containing these vertices are the basis elements.*

Proof. Fix an isomorphism $\phi:\Gamma\to\mathbf{H}(\mathbb{P},\mathbb{H})$. By Proposition 3.8, we can as well argue with exterior points of $\mathbf{H}(\mathbb{P},\mathbb{H})$. Let x_i, for $i=1,\dots,m$, be exterior points forming a (projective) basis for $\phi(U)$. Let K be a complement to $\phi(U)$ in $\mathbb{P}$, which is the intersection of hyperplanes in $\mathbb{H}$. Notice that such subspace K exists as $\bigcap_{H\in\mathbb{H}}H$ is empty. Moreover, as K has finite codimension in V, all hyperplanes of V containing K are in $\mathbb{H}$. If for each $i\in\{1,\dots,m\}$ we have $x_i=\{(p_i,H)\in\mathbf{H}(\mathbb{P},\mathbb{H})\mid H\in\mathbb{H}\}$, then the vertices $(p_i,\langle K,\{p_j\mid j\in\{1,\dots,m\}\setminus\{i\}\}\rangle)\in x_i$, with $i=1,\dots,m$, form the clique we are looking for. ∎

Notation 3.13. Let $n\geq 3$. For a vertex $\mathbf{x}$ of $\Gamma\cong\mathbf{H}(\mathbb{P},\mathbb{H})$, we write $\approx_{\mathbf{x}}$ for the relation $\approx$ defined on $\mathbf{x}^\perp$ (bear in mind that the latter is isomorphic to $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$ by Proposition 2.2, where $\mathbb{P}_0$ and $\mathbb{H}_0$ are hyperplanes of $\mathbb{P}$ and $\mathbb{H}$, respectively).

Lemma 3.14. *Let $n\geq 3$. Let $\mathbf{x}$ be a vertex of Γ. Then $\approx_{\mathbf{x}}$ is the restriction of $\approx$ to $\mathbf{x}^\perp$.*

In particular, if p is an interior point of Γ with $p\cap\mathbf{x}^\perp\neq\emptyset$, then $p\cap\mathbf{x}^\perp$ is an interior point or an interior hyperplane of $\mathbf{x}^\perp$, and conversely, if q is an interior point of $\mathbf{x}^\perp$, then there exists an interior point or hyperplane q' of Γ with $q'\cap\mathbf{x}^\perp=q$.

Proof. Fix an isomorphism $\phi:\Gamma\to\mathbf{H}$. As above, we argue in $\mathbf{H}$ rather than in Γ. Let $\phi(\mathbf{x})=(x,X)$. Now the statement follows from the fact that, for $\mathbf{a},\mathbf{b}\in\mathbf{x}^\perp$, with $\mathbf{a}\approx_{\mathbf{x}}\mathbf{b}$ and $\phi(\mathbf{a})=(a,A)$, $\phi(\mathbf{b})=(b,B)$, the statements $A\cap X=B\cap X$ and $A=B$ are equivalent. ∎

Notation 3.15. In view of the lemma, we can choose the equivalence relation $\approx_{\mathbf{x}}^p$ on $\mathbf{x}^\perp$ in such a way that $(\approx_{\mathbf{x}})^p=(\approx^p)_{\mathbf{x}}$. In that case, there is no harm in writing $\approx^p_{\mathbf{x}}$ to denote this relation. In particular, there is an injective map from the set of interior points of $\mathbf{x}^\perp$ into the set of interior points of Γ.

Lemma 3.16. *Let $n \geq 3$ and let $\mathbf{x}$ be a vertex of Γ. Then the interior projective space on $\mathbf{x}^\perp$ is a hyperplane of the interior projective space on Γ.*

Proof. Fix an isomorphism $\Gamma \cong \mathbf{H}(\mathbb{P},\mathbb{H})$. By Proposition 3.8 this isomorphism of graphs induces an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{H}(\mathbb{P},\mathbb{H})$. The vertex $\mathbf{x} \in \Gamma$ is mapped onto a non-incident point-hyperplane pair of $\mathbf{H}(\mathbb{P},\mathbb{H})$, say (x, X). The neighbors of $\mathbf{x}$ are mapped onto point-hyperplane pairs (y, Y) with $y \in X$, inducing a map of the set of interior points of Γ that meet $\mathbf{x}^\perp$ non-trivially onto the set of exterior points of $\mathbf{H}(\mathbb{P},\mathbb{H})$ that intersect $(x, X)^\perp$ non-trivially. But that set of exterior points form a hyperplane of the exterior projective space on $\mathbf{H}(\mathbb{P},\mathbb{H})$, and the lemma is proved. ∎

4. Locally point-hyperplane graphs

Throughout the whole section, we take $n \geq 3$, and Γ a connected, locally $\mathbf{H}(\mathbb{P},\mathbb{H})$ graph for some projective space $\mathbb{P}$ of dimension n (possibly infinite) and subspace $\mathbb{H}$ of $\mathbb{P}^{\mathrm{dual}}$ with trivial annihilator in $\mathbb{P}$. Thus, the fact that Γ is locally $\mathbf{H}(\mathbb{P},\mathbb{H})$ means that, for each vertex $\mathbf{x}$ of Γ, there is an isomorphism $\mathbf{x}^\perp \to \mathbf{H}(\mathbb{P},\mathbb{H})$. Consequently, by Corollary 3.9, the interior projective space on $\mathbf{x}^\perp$ is isomorphic to $\mathbb{P}$ or $\mathbb{H}$. The goal of this section is, by use of these isomorphisms, to show that Γ is isomorphic to the non-incident point-hyperplane graph $\mathbf{H}(\mathbb{P}_1,\mathbb{H}_1)$ for some projective space $\mathbb{P}_1$ and subspace $\mathbb{H}_1$ of ${\mathbb{P}_1}^{\mathrm{dual}}$. This will establish Theorem 1.1.

Notice that the definitions of interior points and lines are only local and may differ on different perps. It is one task of this section to show that there is a well-defined notion of *global points* and *global lines* on the whole graph. To avoid confusion, we will index each interior point p and each interior line l by the vertex $\mathbf{x}$ whose perp it belongs to, so we write $p_{\mathbf{x}}$ and $l_{\mathbf{x}}$ instead of p and l. These interior points and lines are called *local points* and *local lines*, respectively. We do the same for the relations $\approx$, $\approx^p$, $\approx^h$ obtaining the local relations $\approx_{\mathbf{x}}$, $\approx^p_{\mathbf{x}}$, $\approx^h_{\mathbf{x}}$.

Lemma 4.1. *Let $\mathbf{x}$ and $\mathbf{y}$ be two adjacent vertices of Γ. Then there is a choice of local equivalence relations $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ such that the restrictions of $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.*

Proof. This follows immediately from a repeated application of Lemma 3.14 to $\mathbf{x}^\perp \cong \mathbf{H}(\mathbb{P},\mathbb{H})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and to $\mathbf{y}^\perp \cong \mathbf{H}(\mathbb{P},\mathbb{H})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp$. ∎

The preceding lemma allows us to transfer points from $\mathbf{x}^\perp$ to $\mathbf{y}^\perp$. Indeed, if there is a local point $p_\mathbf{x}$ in $\mathbf{x}^\perp$ that lies in the hyperplane $Y_\mathbf{x}$ induced by the vertex $\mathbf{y}$ on $\mathbf{x}^\perp$, the point $p_\mathbf{x}$ corresponds to a point $p_\mathbf{y}$ of $\mathbf{y}^\perp$. That point $p_\mathbf{y}$ is simply the $\approx^p_\mathbf{y}$ equivalence class that contains the set $p_\mathbf{x} \cap \mathbf{y}^\perp$. We shall write $\pi^\mathbf{y}_\mathbf{x}(p_\mathbf{x})$ to denote $p_\mathbf{y}$.

In the next three lemmas we prove some technical statements enabling us to prove simple connectedness of Γ. (A graph is simply connected if it is connected and every cycle in it can be triangulated.)

Lemma 4.2. *For mutually adjacent vertices* $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ *of* Γ, *the assignment* π *has the following properties.*

(i) $\pi^\mathbf{x}_\mathbf{y}\pi^\mathbf{y}_\mathbf{x}(p_\mathbf{x}) = p_\mathbf{x}$.
(ii) If $p_\mathbf{x}$ *is a local point at* $\mathbf{x}$ *containing* $\mathbf{y}$ *and such that* $\pi^\mathbf{y}_\mathbf{x}(p_\mathbf{x})$ *contains* $\mathbf{z}$, *then* $\pi^\mathbf{z}_\mathbf{y}\pi^\mathbf{y}_\mathbf{x}(p_\mathbf{x})$ *contains* $\mathbf{x}$ *and* $\pi^\mathbf{x}_\mathbf{z}\pi^\mathbf{z}_\mathbf{y}\pi^\mathbf{y}_\mathbf{x}(p_\mathbf{x}) = p_\mathbf{x}$.
(iii) If $\mathbf{z} = (z_\mathbf{x}, Z_\mathbf{x})$ *in* $\mathbf{x}^\perp$ *and* $\mathbf{z} = (z_\mathbf{y}, Z_\mathbf{y})$ *in* $\mathbf{y}^\perp$, *then* $z_\mathbf{y} = \pi^\mathbf{y}_\mathbf{x}(z_\mathbf{x})$.

Proof. (i) is obvious.

(ii). Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ is a triangle. In view of Lemma 4.1, we may assume that $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{y}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and that $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{z}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{z}^\perp$. Let $p_\mathbf{x}$ be an interior point of $\mathbf{x}^\perp$ such that $p_\mathbf{x} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. By analysis of $\mathbf{x}^\perp$, we can find two vertices, say $\mathbf{u}$ and $\mathbf{v}$, in $p_\mathbf{x} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. Now the above choices of local equivalence relations imply that $(\mathbf{u},\mathbf{v})$ belongs to $\approx^p_\mathbf{y} \cap \approx^p_\mathbf{z}$ (indeed, $(\mathbf{u},\mathbf{v})$ belongs to both $\approx^p_\mathbf{x} \cap \approx^p_\mathbf{y}$ and $\approx^p_\mathbf{x} \cap \approx^p_\mathbf{z}$). By Lemma 3.2 this forces that $\approx^p_\mathbf{y}$ and $\approx^p_\mathbf{z}$ have the same restriction to $\mathbf{y}^\perp \cap \mathbf{z}^\perp$.

(iii). Let $\mathbf{y} = (y_\mathbf{x}, Y_\mathbf{x})$ in $\mathbf{x}^\perp$. Then $z_\mathbf{x} \in Y_\mathbf{x}$ and $y_\mathbf{x} \in Z_\mathbf{x}$. In $\mathbf{x}^\perp$, we see at least two hyperplanes Z' and Z'' that contain $y_\mathbf{x}$ but not $z_\mathbf{x}$. Put $\mathbf{z}' = (z_\mathbf{x}, Z')$ and $\mathbf{z}'' = (z_\mathbf{x}, Z'')$. Then $\mathbf{z}$, $\mathbf{z}'$, $\mathbf{z}''$ belong to the same $\approx^p_\mathbf{y}$-class, namely $\pi^\mathbf{y}_\mathbf{x}(z_\mathbf{x})$. Hence $z_\mathbf{y} = \pi^\mathbf{y}_\mathbf{x}(z_\mathbf{x})$. ∎

Lemma 4.3. *Let* $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ *be a path of vertices in* Γ. *Then for* $\mathbf{x} = (x_\mathbf{y}, X_\mathbf{y})$ *and* $\mathbf{z} = (z_\mathbf{y}, Z_\mathbf{y})$ *inside* $\mathbf{y}^\perp$, *if* $X_\mathbf{y} \cap Z_\mathbf{y} \cap x_\mathbf{y} z_\mathbf{y} = \emptyset$ *or if* $X_\mathbf{y} = Z_\mathbf{y}$, *there is a path of vertices in* $\mathbf{x}^\perp \cap \mathbf{z}^\perp$ *from* $\mathbf{y}$ *to a vertex in* $\{\mathbf{w},\mathbf{x},\mathbf{z}\}^\perp$.

Notice that, for example, we have $X_\mathbf{y} \cap Z_\mathbf{y} \cap \langle x_\mathbf{y}, z_\mathbf{y} \rangle = \emptyset$, in case $x_\mathbf{y} = z_\mathbf{y}$.

Proof. Choose local equivalence relations $\approx^p_\mathbf{w}$, $\approx^p_\mathbf{x}$, $\approx^p_\mathbf{y}$, and $\approx^p_\mathbf{z}$ such that $\approx^p_\mathbf{w}$ and $\approx^p_\mathbf{x}$ coincide on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$, such that $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{y}$ coincide on $\mathbf{x}^\perp \cap \mathbf{y}^\perp$, and such that $\approx^p_{\mathbf{y}}$ and $\approx^p_{\mathbf{z}}$ coincide on $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ as indicated in Lemma 4.1. Application of Lemma 3.16 to the interior projective space of $\mathbf{y}^\perp \cong \mathbf{H}(\mathbb{P},\mathbb{H})$ shows that the interior projective spaces of $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and of $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ correspond to hyperplanes of $\mathbf{y}^\perp \cong \mathbf{H}(\mathbb{P},\mathbb{H})$. We have to investigate $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. We have $\mathbf{x} = (x_{\mathbf{y}}, X_{\mathbf{y}})$ and $\mathbf{z} = (z_{\mathbf{y}}, Z_{\mathbf{y}})$ inside $\mathbf{y}^\perp$. Then the graph $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ (considered inside $\mathbf{y}^\perp$) consists of the non-incident point-hyperplane pairs whose points are contained in $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and whose hyperplanes contain the subspace $\langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle$.

First, let us assume $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle = \emptyset$. Also assume that $x_{\mathbf{y}} \neq z_{\mathbf{y}}$ and denote the intersection $x_{\mathbf{y}} z_{\mathbf{y}} \cap X_{\mathbf{y}}$ by $a_{\mathbf{y}}$. Inside $\mathbf{x}^\perp$ denote $\mathbf{w}$ by $(w_{\mathbf{x}}, W_{\mathbf{x}})$ and $\mathbf{y}$ by $(y_{\mathbf{x}}, Y_{\mathbf{x}})$. Consider $\mathbf{x}^\perp$, in which the point $a_{\mathbf{y}} \in X_{\mathbf{y}}$ arises as $a_{\mathbf{x}}$ inside $Y_{\mathbf{x}}$. Inside $\mathbf{y}^\perp$, the intersection $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ contains a line $l_{\mathbf{y}}$. This line $l_{\mathbf{y}}$ arises as a subspace $l_{\mathbf{x}}$ of $\mathbf{x}^\perp$ that is contained in $Y_{\mathbf{x}}$. As there exists a $\mathbf{y}'$ in $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$, we can assume, up to a change of $\mathbf{y}$ into $\mathbf{y}'$, that $w_{\mathbf{x}}$ is also contained in $Y_{\mathbf{x}}$. (Indeed, choose a hyperplane $H_{\mathbf{x}}$ that contains $a_{\mathbf{x}}$, $w_{\mathbf{x}}$, and $y_{\mathbf{x}}$ but not $l_{\mathbf{x}}$, and choose a point $p_{\mathbf{x}}$ on $l_{\mathbf{x}}$ off $H_{\mathbf{x}}$. The vertex $(p_{\mathbf{x}}, H_{\mathbf{x}})$ gives rise to a vertex $\mathbf{y}'$ that is adjacent to $\mathbf{x}$ and $\mathbf{y}$. Local analysis of $\mathbf{y}^\perp$ shows that the hyperplane of the vertex $\mathbf{y}'$ contains the point $x_{\mathbf{y}}$ and the point $a_{\mathbf{y}}$, whence also the point $z_{\mathbf{y}}$. Moreover, the point of $\mathbf{y}'$ is contained in $l_{\mathbf{y}}$, whence also in $Z_{\mathbf{y}}$, and $\mathbf{y}'$ is a neighbor of $\mathbf{z}$.) Inside $\mathbf{x}^\perp$ we have now the following setting. The hyperplane $Y_{\mathbf{x}}$ contains the points $w_{\mathbf{x}}$ and $a_{\mathbf{x}}$ as well as the line $l_{\mathbf{x}}$. Note that $l_{\mathbf{x}}$ has to intersect the hyperplane $W_{\mathbf{x}}$. If $\langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ does not intersect $l_{\mathbf{x}} \cap W_{\mathbf{x}}$, then we can choose a point inside $l_{\mathbf{x}} \cap W_{\mathbf{x}}$ and a non-incident hyperplane that contains $\langle a_{\mathbf{x}}, w_{\mathbf{x}}, y_{\mathbf{x}} \rangle$, yielding a vertex that is adjacent to $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and – after local analysis of $\mathbf{y}^\perp$ – also to $\mathbf{z}$. Therefore assume that $\langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ does intersect $l_{\mathbf{x}} \cap W_{\mathbf{x}}$. Then fix the point $u_{\mathbf{x}} := \langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle \cap l_{\mathbf{x}} \cap W_{\mathbf{x}}$ and choose a hyperplane $U_{\mathbf{x}}$ that contains $a_{\mathbf{x}}$ and $y_{\mathbf{x}}$ but not $u_{\mathbf{x}}$. The pair $(u_{\mathbf{x}}, U_{\mathbf{x}})$ describes another vertex, $\mathbf{u}$ say, that is adjacent to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Inside $\mathbf{u}^\perp$ we have a hyperplane $X_{\mathbf{u}}$ of $\mathbf{x}$, a line $k_{\mathbf{u}}$ in $X_{\mathbf{u}}$ that arises from a line $k_{\mathbf{x}}$ contained in the intersection $U_{\mathbf{x}} \cap W_{\mathbf{x}}$ of the hyperplanes of the vertices $\mathbf{u}$ and $\mathbf{w}$ inside $\mathbf{x}^\perp$, and the hyperplane $Z_{\mathbf{u}}$ of $\mathbf{z}$. Choose a point $v_{\mathbf{u}}$ in $k_{\mathbf{u}} \cap Z_{\mathbf{u}}$ and a hyperplane $V_{\mathbf{u}}$ on $x_{\mathbf{u}} z_{\mathbf{u}}$ that does not contain $v_{\mathbf{u}}$. Obviously, this vertex $\mathbf{v} = (v_{\mathbf{u}}, V_{\mathbf{u}})$ is adjacent to $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{z}$. In $\mathbf{x}^\perp$, however, we see $\mathbf{v}$ as $(v_{\mathbf{x}}, V_{\mathbf{x}})$ whose hyperplane $V_{\mathbf{x}}$ contains the points $a_{\mathbf{x}}$ and $u_{\mathbf{x}}$, therefore also $w_{\mathbf{x}}$. Moreover, $v_{\mathbf{x}}$ is contained in $k_{\mathbf{x}}$, whence also in $W_{\mathbf{x}}$, and $\mathbf{v}$ is the required vertex.

If $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle = \emptyset$ and $x_{\mathbf{y}} = z_{\mathbf{y}}$, then similar but simpler arguments yield a proof. Also, the case where $X_{\mathbf{y}} = Z_{\mathbf{y}}$ runs along the same lines and is easier to prove. ∎

Lemma 4.4. *For every path* $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ *in* Γ *there is a vertex* $\mathbf{x}_0 \in \{\mathbf{x}\} \cup \{\mathbf{w}, \mathbf{x}, \mathbf{y}\}^{\perp}$ *such that, with* $\mathbf{x}_0 = (x^0_{\mathbf{y}}, X^0_{\mathbf{y}})$ *and* $\mathbf{z} = (z_{\mathbf{y}}, Z_{\mathbf{y}})$ *inside* $\mathbf{y}^{\perp}$, *we have* $\langle x^0_{\mathbf{y}}, z_{\mathbf{y}} \rangle \cap X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} = \emptyset$.

Proof. Choose a path $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ of vertices in Γ, and fix local equivalence relations $\approx^p_{\mathbf{w}}$, $\approx^p_{\mathbf{x}}$, $\approx^p_{\mathbf{y}}$, and $\approx^p_{\mathbf{z}}$ as in the proof of the preceding lemma. Inside $\mathbf{y}^{\perp}$, let $\mathbf{x}$ correspond to $(x_{\mathbf{y}}, X_{\mathbf{y}})$ and $\mathbf{z}$ correspond to $(z_{\mathbf{y}}, Z_{\mathbf{y}})$. Suppose that $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle \neq \emptyset$. Then $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle$ is a point; $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \backslash x_{\mathbf{y}} z_{\mathbf{y}}$ contains (the point set of) an affine line, for $n = 3$, and (the point set of) a dual affine plane, for $n \geq 4$. The set of common neighbors of $\mathbf{x}$ and $\mathbf{z}$ in $\mathbf{y}^{\perp}$ corresponds to the set of all non-incident point-hyperplane pairs $(p_{\mathbf{y}}, H_{\mathbf{y}})$ with $p_{\mathbf{y}} \in X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and $H_{\mathbf{y}} \supset x_{\mathbf{y}} z_{\mathbf{y}}$. This implies that for any point $p_{\mathbf{y}} \in X_{\mathbf{y}} \cap Z_{\mathbf{y}} \backslash x_{\mathbf{y}} z_{\mathbf{y}}$ we can find a vertex $(p_{\mathbf{y}}, H_{\mathbf{y}})$ in $\mathbf{y}^{\perp}$ adjacent to both $\mathbf{x}$ and $\mathbf{z}$. Now consider $\mathbf{x}^{\perp}$. Let $\mathbf{w} = (w_{\mathbf{x}}, W_{\mathbf{x}})$ and $\mathbf{y} = (y_{\mathbf{x}}, Y_{\mathbf{x}})$. Any vertex $\mathbf{x}_0 = (x^0_{\mathbf{x}}, X^0_{\mathbf{x}})$ adjacent to $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ consists of a point $x^0_{\mathbf{x}} \in W_{\mathbf{x}} \cap Y_{\mathbf{x}}$ and a non-incident hyperplane $X^0_{\mathbf{x}} \supset w_{\mathbf{x}} y_{\mathbf{x}}$. Hence, as above in $\mathbf{y}^{\perp}$, we can choose $x^0_{\mathbf{x}}$ freely on an affine line for $n = 3$ or a dual affine plane for $n \geq 4$. This translates to $\mathbf{y}^{\perp}$ as follows. The line $w_{\mathbf{x}} y_{\mathbf{x}}$ intersects $Y_{\mathbf{x}}$ in a point, $a_{\mathbf{x}}$ say, which gives rise to a point $a_{\mathbf{y}} \in X_{\mathbf{y}}$ of $\mathbf{y}^{\perp}$. So all these hyperplanes $X^0_{\mathbf{x}}$ arise as hyperplanes $X^0_{\mathbf{y}}$ in $\mathbf{y}^{\perp}$ that contain the line $x_{\mathbf{y}} a_{\mathbf{y}}$. Notice that this line $x_{\mathbf{y}} a_{\mathbf{y}}$ is the largest subspace of $\mathbf{y}^{\perp}$ that is contained in all these hyperplanes $X^0_{\mathbf{y}}$. If for some fixed choice of $x^0_{\mathbf{y}}$, there exists a hyperplane $X^0_{\mathbf{y}}$ of $\mathbf{y}^{\perp}$ such that $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x^0_{\mathbf{y}}, z_{\mathbf{y}} \rangle = \emptyset$, we are done. Hence, for a fixed $x^0_{\mathbf{y}}$, suppose all choices for $X^0_{\mathbf{y}}$ contain the point $\langle x^0_{\mathbf{y}}, z_{\mathbf{y}} \rangle \cap Z_{\mathbf{y}}$. But in this case, we can choose another $x^1_{\mathbf{y}}$ instead of $x^0_{\mathbf{y}}$ and find an $X^0_{\mathbf{y}}$ with $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x^1_{\mathbf{y}}, z_{\mathbf{y}} \rangle = \emptyset$. For, suppose for a choice $x^1_{\mathbf{y}}$ distinct from $x^0_{\mathbf{y}}$ still $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x^1_{\mathbf{y}}, z_{\mathbf{y}} \rangle \neq \emptyset$ for all possible $X^0_{\mathbf{y}}$ inside $\mathbf{y}^{\perp}$. Then the points $u_{\mathbf{y}} := \langle x^0_{\mathbf{y}}, z_{\mathbf{y}} \rangle \cap Z_{\mathbf{y}}$ and $v_{\mathbf{y}} := \langle x^1_{\mathbf{y}}, z_{\mathbf{y}} \rangle \cap Z_{\mathbf{y}}$ span a line as $z_{\mathbf{y}} \notin Z_{\mathbf{y}}$. But this line $u_{\mathbf{y}} v_{\mathbf{y}}$ has to coincide with the line $x_{\mathbf{y}} a_{\mathbf{y}}$. In particular, $x_{\mathbf{y}}$ is contained in $Z_{\mathbf{y}}$. But this contradicts our assumption that $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x_{\mathbf{y}}, z_{\mathbf{y}} \rangle \neq \emptyset$. Hence we can find an $x^1_{\mathbf{y}} \notin X^0_{\mathbf{y}}$ with $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap \langle x^1_{\mathbf{y}}, z_{\mathbf{y}} \rangle = \emptyset$, and so the vertex $(x^1_{\mathbf{y}}, X^0_{\mathbf{y}})$ is as required. ∎

We owe the following proposition to Andries Brouwer, who observed that the combination of the two preceding lemmas yields simple connectedness.

Proposition 4.5. *The graph* Γ, *considered as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected. Moreover, the diameter of* Γ *equals two.*

Proof. Lemma 4.4 shows that for every path of distinct vertices $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in Γ there exists a vertex $\mathbf{x}_0 \in \{\mathbf{w}, \mathbf{x}, \mathbf{y}\}^{\perp}$ with $\mathbf{x}_0 = (x^0_{\mathbf{y}}, X^0_{\mathbf{y}})$ in $\mathbf{y}^{\perp}$ such that $\langle x^0_{\mathbf{y}}, z_{\mathbf{y}}\rangle \cap X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} = \emptyset$. Lemma 4.3, on the other hand, implies that there exists a path of vertices inside $\mathbf{x}_0^{\perp} \cap \mathbf{z}^{\perp}$ from $\mathbf{y}$ to a vertex $\mathbf{v}$ that is adjacent to $\mathbf{w}$, $\mathbf{x}_0$, and $\mathbf{z}$. Simple connectedness of Γ follows.

As for the second statement, suppose $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ is a path in Γ, then by the above arguments there is a vertex $\mathbf{v}$ in $\mathbf{w}^{\perp} \cap \mathbf{z}^{\perp}$. Hence $\mathbf{z}$ is at distance at most two from $\mathbf{w}$. This implies that the diameter of Γ is at most two and settles the proof of the proposition. ∎

Lemma 4.6. *There is a choice of local equivalence relations $\approx^p_{\mathbf{x}}$ for all $\mathbf{x} \in \Gamma$ such that, for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$, the restrictions of $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ to $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp}$ coincide.*

Proof. Since Γ is simply connected (by Proposition 4.5), the lemma follows immediately from the triangle analysis of Lemma 4.2. ∎

Notation 4.7. Fix a choice of $\approx^p_{\mathbf{x}}$, for all vertices $\mathbf{x}$ of Γ, as in Lemma 4.6 and set $\approx^p = \bigcup_{\mathbf{x} \in \Gamma} \approx^p_{\mathbf{x}}$.

Lemma 4.8. *Let $\mathbf{x}$ and $\mathbf{y}$ be vertices of Γ such that $\mathbf{x} \approx^p_{\mathbf{u}} \mathbf{y}$ for some vertex $\mathbf{u}$ in $\{\mathbf{x},\mathbf{y}\}^{\perp}$. Then $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is connected and $\mathbf{x} \approx^p_{\mathbf{v}} \mathbf{y}$ for every vertex $\mathbf{v}$ in $\{\mathbf{x},\mathbf{y}\}^{\perp}$.*

Proof. Let $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$ be as in the hypothesis and let $\mathbf{v} \in \{\mathbf{x},\mathbf{y}\}^{\perp}$ be an additional vertex. If $\mathbf{u} \perp \mathbf{v}$, then the claim is true by Lemma 4.6.

Thus, it is sufficient to show that the induced subgraph $\{\mathbf{x},\mathbf{y}\}^{\perp}$ of Γ is connected. In $\mathbf{x}^{\perp}$ we have $\mathbf{u} = (u_{\mathbf{x}}, U_{\mathbf{x}})$ and $\mathbf{v} = (v_{\mathbf{x}}, V_{\mathbf{x}})$. Moreover, the intersection $X_{\mathbf{u}} \cap Y_{\mathbf{u}}$ from $\mathbf{u}^{\perp}$ arises as a hyperplane $W_{\mathbf{x}} = \pi^{\mathbf{x}}_{\mathbf{u}}(X_{\mathbf{u}} \cap Y_{\mathbf{u}})$ of $U_{\mathbf{x}}$ in $\mathbf{x}^{\perp}$. Therefore the intersection $W_{\mathbf{x}} \cap V_{\mathbf{x}}$ contains a point $p_{\mathbf{x}}$. If in $\mathbf{x}^{\perp}$ the line $u_{\mathbf{x}}v_{\mathbf{x}}$ does not contain $p_{\mathbf{x}}$, we can find a hyperplane $H_{\mathbf{x}} \supset u_{\mathbf{x}}v_{\mathbf{x}}$ that does not contain $p_{\mathbf{x}}$, and $(p_{\mathbf{x}}, H_{\mathbf{x}})$ is a vertex of $\mathbf{x}^{\perp}$ which is adjacent to both $\mathbf{u}$ and $\mathbf{v}$. But inside $\mathbf{u}^{\perp}$ this vertex also corresponds to some point-hyperplane pair, whose point is contained in $Y_{\mathbf{u}}$ and whose hyperplane contains $y_{\mathbf{u}} = x_{\mathbf{u}}$. In particular, this vertex is also adjacent to $\mathbf{y}$, and we are done.

So assume we have $p_{\mathbf{x}} \in u_{\mathbf{x}}v_{\mathbf{x}}$ in $\mathbf{x}^{\perp}$. Then choose any hyperplane $H_{\mathbf{x}}$ that contains $u_{\mathbf{x}}$ but not $p_{\mathbf{x}}$. Then the vertex $\mathbf{t} := (p_{\mathbf{x}}, H_{\mathbf{x}})$ is adjacent to $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}$, but not $\mathbf{v}$. Inside $\mathbf{t}^{\perp}$ we have hyperplanes $X_{\mathbf{t}}$ and $Y_{\mathbf{t}}$ coming from $\mathbf{x}$ and $\mathbf{y}$. The intersection $X_{\mathbf{t}} \cap Y_{\mathbf{t}}$ corresponds to a subspace $S_{\mathbf{x}}$ of $H_{\mathbf{x}}$ (the hyperplane of the vertex $\mathbf{t}$) in $\mathbf{x}^{\perp}$. The intersection $S_{\mathbf{x}} \cap V_{\mathbf{x}}$ in $\mathbf{x}^{\perp}$ contains some point $q_{\mathbf{x}}$. If $q_{\mathbf{x}}$ lies on the line $p_{\mathbf{x}}v_{\mathbf{x}}$, then $q_{\mathbf{x}} = p_{\mathbf{x}}v_{\mathbf{x}} \cap H_{\mathbf{x}} = p_{\mathbf{x}}u_{\mathbf{x}} \cap H_{\mathbf{x}} = u_{\mathbf{x}}$, and we have $u_{\mathbf{x}} \in V_{\mathbf{x}}$. But this contradicts $p_{\mathbf{x}} \in u_{\mathbf{x}}v_{\mathbf{x}}$, as $p_{\mathbf{x}} \in V_{\mathbf{x}} \cap U_{\mathbf{x}}$, $v_{\mathbf{x}} \notin V_{\mathbf{x}}$ and $u_{\mathbf{x}} \in V_{\mathbf{x}} \setminus U_{\mathbf{x}}$, whence $u_{\mathbf{x}}v_{\mathbf{x}} \cap V_{\mathbf{x}} \cap U_{\mathbf{x}} = \emptyset$. Therefore we have $q_{\mathbf{x}} \notin p_{\mathbf{x}}v_{\mathbf{x}}$ and we are in the situation of the preceding paragraph with the vertex $\mathbf{t}$ instead of $\mathbf{u}$. ∎

Lemma 4.9. *The relation $\approx^p$ on the vertices of Γ is an equivalence relation.*

Proof. Reflexivity and symmetry follow from reflexivity and symmetry of each $\approx^p_{\mathbf{x}}$. To prove transitivity, assume that $\mathbf{x} \approx^p \mathbf{y}$ and $\mathbf{y} \approx^p \mathbf{z}$. Let $\mathbf{a}$ be a point in $\{\mathbf{x},\mathbf{y}\}^\perp$. Then, by Lemma 4.8, we find $\mathbf{x} \approx^p_{\mathbf{a}} \mathbf{y}$. In $\mathbf{a}^\perp$, we have $\mathbf{x} = (x_{\mathbf{a}}, X_{\mathbf{a}})$ and $\mathbf{y} = (x_{\mathbf{a}}, Y_{\mathbf{a}})$. A vertex $\mathbf{b} \in \{\mathbf{x},\mathbf{y},\mathbf{a}\}^\perp$ corresponds in $\mathbf{a}^\perp$ to a pair $(b_{\mathbf{a}}, B_{\mathbf{a}})$ with $b_{\mathbf{a}} \in X_{\mathbf{a}} \cap Y_{\mathbf{a}}$ and $x_{\mathbf{a}} = y_{\mathbf{a}} \in B_{\mathbf{a}}$. Inside $\mathbf{y}^\perp$, the points $\mathbf{a}$ and $\mathbf{b}$ correspond to pairs $(a_{\mathbf{y}}, A_{\mathbf{y}})$ and $(b_{\mathbf{y}}, B_{\mathbf{y}})$, respectively, with $b_{\mathbf{y}} \in S^A_{\mathbf{y}}$ and $a_{\mathbf{y}} \in S^B_{\mathbf{y}}$, where $S^A_{\mathbf{y}} = \pi^{\mathbf{y}}_{\mathbf{a}}(X_{\mathbf{a}} \cap Y_{\mathbf{a}})$ and $S^B_{\mathbf{y}} = \pi^{\mathbf{y}}_{\mathbf{b}}(X_{\mathbf{b}} \cap Y_{\mathbf{b}})$, which are hyperplanes of $A_{\mathbf{y}}$ and $B_{\mathbf{y}}$, respectively. If we let $\mathbf{b}$ vary in $\{\mathbf{x},\mathbf{y},\mathbf{a}\}^\perp$, we obtain all points corresponding to pairs $(b_{\mathbf{y}}, B_{\mathbf{y}})$ with $b_{\mathbf{y}} \in S^A_{\mathbf{y}}$ and $a_{\mathbf{y}} \in B_{\mathbf{y}}$.

The above and Lemma 4.8 imply that for any point $\mathbf{c} = (c_{\mathbf{y}}, C_{\mathbf{y}})$ in $\{\mathbf{x},\mathbf{y}\}^\perp$, there exists a unique hyperplane $S^C_{\mathbf{y}}$ of $C_{\mathbf{y}}$ such that the points $\mathbf{p}$ in $\{\mathbf{x},\mathbf{y},\mathbf{c}\}^\perp$ are precisely the points $(p_{\mathbf{y}}, P_{\mathbf{y}})$ from $\mathbf{y}^\perp$ with $p_{\mathbf{y}} \in S^C_{\mathbf{y}}$ and $P_{\mathbf{y}}$ containing $c_{\mathbf{y}}$ but not $p_{\mathbf{y}}$.

Since $n \geq 3$, we can find adjacent points $\mathbf{d} = (d_{\mathbf{y}}, D_{\mathbf{y}})$ and $\mathbf{e} = (e_{\mathbf{y}}, E_{\mathbf{y}})$ in $\{\mathbf{x},\mathbf{y}\}^\perp$ with $d_{\mathbf{y}}, e_{\mathbf{y}} \in S^A_{\mathbf{y}}$ and $a_{\mathbf{y}} \in D_{\mathbf{y}} \cap E_{\mathbf{y}}$. So, $\mathbf{d},\mathbf{e}$ are in $\mathbf{a}^\perp$. Applying the above with respect to the point $\mathbf{e}$, we see that the line on $a_{\mathbf{y}}$ and $d_{\mathbf{y}}$ is contained in $S^E_{\mathbf{y}}$. So, by varying $e_{\mathbf{y}}$ over $S^A_{\mathbf{y}} \setminus \{d_{\mathbf{y}}\}$, we find that all points $(d_{\mathbf{y}}, D'_{\mathbf{y}})$ with $D'_{\mathbf{y}}$ an arbitrary hyperplane in $\mathbb{H}$ are in $\{\mathbf{x},\mathbf{y}\}^\perp$. Varying $\mathbf{d}$ and $\mathbf{e}$ we find that all points $\mathbf{p} = (p_{\mathbf{y}}, P_{\mathbf{y}})$, with $p_{\mathbf{y}}$ in the subspace spanned by $a_{\mathbf{y}}$ and $S^A_{\mathbf{y}}$ and $P_{\mathbf{y}}$ an arbitrary hyperplane in $\mathbb{H}$ not on $p_{\mathbf{y}}$, are in $\{\mathbf{x},\mathbf{y}\}^\perp$.

Similarly, we find that for some hyperplane H in $\mathbb{H}$ all points $\mathbf{q} = (q_{\mathbf{y}}, Q_{\mathbf{y}})$ with $q_{\mathbf{y}} \in H$ are in $\{\mathbf{z},\mathbf{y}\}^\perp$. Hence, as $n \geq 3$, there is a point $\mathbf{r} = (r_{\mathbf{y}}, R_{\mathbf{y}})$ in $\{\mathbf{x},\mathbf{y},\mathbf{z}\}^\perp$. Then $\mathbf{x} \approx^p_{\mathbf{r}} \mathbf{y}$ and $\mathbf{y} \approx^p_{\mathbf{r}} \mathbf{z}$, see Lemma 4.8, and inside $\mathbf{r}^\perp$ we see that $\mathbf{x} \approx^p_{\mathbf{r}} \mathbf{z}$. This proves transitivity of $\approx^p$. ∎

All statements and results about the local relations $\approx^p_{\mathbf{x}}$ are also true for the local relations $\approx^h_{\mathbf{x}}$, and we can define a global relation $\approx^h = \bigcup_{\mathbf{x} \in \Gamma} \approx^h_{\mathbf{x}}$ with the same nice properties on the local intersections. The following notions will be used for the construction of a geometry from Γ.

Definition 4.10. A *global point* of Γ is defined as an equivalence class of $\approx^p$. Dually, define a *global hyperplane* as an equivalence class of $\approx^h$.

If p is a global point and $\mathbf{x}$ a vertex of Γ such that $\mathbf{x}^\perp$ meets p nontrivially, then, by Lemma 4.8, $p \cap \mathbf{x}^\perp$ equals a local point, denoted by $p_{\mathbf{x}}$. Similarly, if H is a global hyperplane meeting $\mathbf{x}^\perp$ nontrivially, then $H \cap \mathbf{x}^\perp$ equals a local hyperplane $H_{\mathbf{x}}$.

We already have a local notion of incidence as defined before Definition 3.6. The global notion of incidence where a global point and global hyperplane are defined to be *incident* if and only if they have empty intersection, extends this local notion as follows from the following lemma.

Lemma 4.11. *Let p be a global point and H a global hyperplane. Then the following statements are equivalent.*

(i) $p \cap H = \emptyset$;
(ii) $p_{\mathbf{x}} \cap H_{\mathbf{x}} = \emptyset$ *for all points* $\mathbf{x}$ *of* Γ *with* $\mathbf{x}^\perp$ *meeting both* p *and* H *nontrivially;*
(iii) $p_{\mathbf{x}} \cap H_{\mathbf{x}} = \emptyset$ *for some point* $\mathbf{x}$ *of* Γ *with* $\mathbf{x}^\perp$ *meeting both* p *and* H *nontrivially.*

Proof. If $p \cap H = \emptyset$, then clearly $p_{\mathbf{x}} \cap H_{\mathbf{x}} = \emptyset$ for all points $\mathbf{x}$ of Γ with $\mathbf{x}^\perp$ meeting both p and H nontrivially. So, (i) implies (ii). Clearly (ii) implies (iii).

Suppose there exists a vertex $\mathbf{y} \in p \cap H$. Then, a vertex $\mathbf{x}$ for which $p_{\mathbf{x}}$ and $H_{\mathbf{x}}$ exist is at distance at most two to $\mathbf{y}$, see Proposition 4.5, and there exists a vertex $\mathbf{z}$ adjacent to both $\mathbf{y}$ and $\mathbf{x}$. The local elements $p_{\mathbf{z}}$ and $H_{\mathbf{z}}$ exist, as $\mathbf{y}$ is a representative of both. But then $p_{\mathbf{x}} \cap z^\perp \neq \emptyset$ as well as $H_{\mathbf{x}} \cap z^\perp \neq \emptyset$. Now inside $\mathbf{x}^\perp \cap \mathbf{z}^\perp$ we see that $p_{\mathbf{x}}$ and $H_{\mathbf{x}}$ have a non-empty intersection. Thus (iii) implies (i). This proves equivalence of the three statements. ∎

Definition 4.12. Let p and q be distinct global points and let $\mathbf{x}$ be a vertex such that $p_{\mathbf{x}}$ and $q_{\mathbf{x}}$ exist. Then the *global line* of Γ spanned by p and q is the set of those global points a such that $a_{\mathbf{x}}$ exists and is contained in the local line $p_{\mathbf{x}} q_{\mathbf{x}}$. Let $\mathbb{P}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$ be the point-line geometry consisting of the point set $\mathcal{P}_\Gamma$ of global points of Γ and the line set $\mathcal{L}_\Gamma$ of global lines of Γ.

Lemma 4.13. *The notion of a global line is well-defined.*

Proof. Let p and q be global points and suppose $\mathbf{x}$ and $\mathbf{y}$ are distinct vertices such that $p_{\mathbf{x}}$, $q_{\mathbf{x}}$, $p_{\mathbf{y}}$, and $q_{\mathbf{y}}$ exist. Suppose r is a global point for which $r_{\mathbf{x}}$ exists and is contained in the local line on $p_{\mathbf{x}}$ and $q_{\mathbf{x}}$. We claim that the local point $r_{\mathbf{y}}$ also exists and is on the local line on $p_{\mathbf{y}}$ and $q_{\mathbf{y}}$.

If $\mathbf{x} \perp \mathbf{y}$, then $p_{\mathbf{x}} \cap p_{\mathbf{y}} \neq \emptyset$ and $q_{\mathbf{x}} \cap q_{\mathbf{y}} \neq \emptyset$, and the claim follows from Lemma 3.16 applied to $\mathbf{x}^\perp$.

Choose vertices $\mathbf{a} \in p_{\mathbf{x}}$, $\mathbf{b} \in q_{\mathbf{x}}$, $\mathbf{c} \in q_{\mathbf{y}}$, and $\mathbf{d} \in p_{\mathbf{y}}$. By Lemma 3.12 we can assume that $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \perp \mathbf{d}$. By Proposition 4.5 there exists a vertex $\mathbf{z}$ adjacent to both $\mathbf{b}$ and $\mathbf{c}$. By Lemma 4.3 (applied to the path $\mathbf{a}$, $\mathbf{b}$, $\mathbf{z}$, $\mathbf{c}$) we can find a vertex $\mathbf{x}_1$ adjacent to $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (indeed, inside $\mathbf{z}^\perp$ we have $b_{\mathbf{z}} = p_{\mathbf{z}} = c_{\mathbf{z}}$. So, the condition of the lemma is satisfied and we can apply that lemma).

As $\{\mathbf{a},\mathbf{b}\}^\perp$ is isomorphic to $\mathbf{H}(\mathbb{P}_0,\mathbb{H}_0)$ for some hyperplane $\mathbb{P}_0$ of $\mathbb{P}$, it is connected, see Proposition 2.3. Thus, by the above, we find that $r_{\mathbf{x}_1}$ exists and is inside the local line on $p_{\mathbf{x}_1}$ and $q_{\mathbf{x}_1}$.

Inside $\mathbf{c}^\perp$ there is a vertex $\mathbf{y}_1$ adjacent to both $\mathbf{d}$ and $\mathbf{x}_1$. Applying the above to the adjacent $\mathbf{x}_1$ and $\mathbf{y}_1$ we find that $r_{\mathbf{y}_1}$ exists and is inside the local line on $p_{\mathbf{y}_1}$ and $q_{\mathbf{y}_1}$. Finally, by connectedness of $\{\mathbf{c},\mathbf{d}\}^\perp$, we can conclude that also $r_{\mathbf{y}}$ exists and is inside the local line on $p_{\mathbf{y}}$ and $q_{\mathbf{y}}$. ∎

Proposition 4.14. *The space $\mathbb{P}_\Gamma$ is a linear space with thick lines.*

Proof. This is an immediate consequence of Lemma 4.13. ∎

As customary in linear spaces, for distinct global points p and q we denote by pq the unique global line on p and q.

Proposition 4.15. *The space $\mathbb{P}_\Gamma$ is a projective space.*

Proof. In view of Proposition 4.14 we only have to verify Pasch's Axiom. Let a, b, c, d be four global points such that ab intersects cd in the global point e. Then $ab = ae$ and $cd = ce$. By Proposition 4.5 and Lemma 3.12, there are vertices $\mathbf{a}$ in a and $\mathbf{e}$ in e such that $\mathbf{a}\perp\mathbf{e}$. Choose a vertex $\mathbf{c}$ in c. Now, by Proposition 4.5, there is a vertex $\mathbf{y}$ adjacent to $\mathbf{e}$ and $\mathbf{c}$. After suitable replacement of $\mathbf{e}$ in e, we can assume that inside $\mathbf{y}^\perp$ we have $\mathbf{c} = (c_{\mathbf{y}}, C_{\mathbf{y}})$ and $\mathbf{e} = (e_{\mathbf{y}}, E_{\mathbf{y}})$ with $C_{\mathbf{y}} \cap E_{\mathbf{y}} \cap \langle c_{\mathbf{y}}, e_{\mathbf{y}}\rangle = \emptyset$. Lemma 4.3 implies the existence of $\mathbf{x} \in \{\mathbf{a},\mathbf{c},\mathbf{e}\}^\perp$. The global lines ae and ce meet $\mathbf{x}^\perp$ in interior lines. In particular, by Pasch's Axiom applied to the interior projective space of $\mathbf{x}^\perp$, there is an interior point $w_{\mathbf{x}}$ on both the interior lines $(ac)_{\mathbf{x}}$ and $(bd)_{\mathbf{x}}$ of $\mathbf{x}^\perp$. Consequently, the global lines ac and bd meet in a global point, whence Pasch's Axiom holds. ∎

We are ready to give a nice description of the hyperplanes of the projective space $\mathbb{P}_\Gamma$ appearing in vertices of Γ. To this end, denote by $\left\langle \mathbf{x}^\perp \right\rangle$ the set of global points that meet $\mathbf{x}^\perp$ in a non-empty set. Notice that this set is a subspace of $\mathbb{P}_\Gamma$; it will turn out to be a hyperplane.

Lemma 4.16. *The set $\left\langle \mathbf{x}^\perp \right\rangle$ does not contain the global point that contains $\mathbf{x}$.*

Proof. Otherwise $\mathbf{x}^\perp$ contains a vertex $\mathbf{y}$ that belongs to the same global point. But then there exists a third vertex $\mathbf{z}$ adjacent to both $\mathbf{x}$ and $\mathbf{y}$, so $\mathbf{x}$ and $\mathbf{y}$ are two adjacent vertices belonging to the same interior point in $\mathbf{z}^\perp$, a contradiction. ∎

Lemma 4.17. *Let* $\mathbf{x},\mathbf{y}\in\Gamma$*. Then* $\mathbf{x}\approx^h\mathbf{y}$ *if and only if* $\left\langle\mathbf{x}^\perp\right\rangle=\left\langle\mathbf{y}^\perp\right\rangle$*.*

Proof. Suppose $\mathbf{x}\approx^h\mathbf{y}$. By symmetry of $\approx^h$ it suffices to show $\left\langle\mathbf{x}^\perp\right\rangle\subseteq\left\langle\mathbf{y}^\perp\right\rangle$. To this end, let $p\in\left\langle\mathbf{x}^\perp\right\rangle$, so that there exists a vertex $\mathbf{p}\in p$ with $\mathbf{p}\perp\mathbf{x}$. By Proposition 4.5, we can find a vertex $\mathbf{z}$ with $\mathbf{x}\perp\mathbf{z}\perp\mathbf{y}$. If $\mathbf{x}=(x_\mathbf{z},X_\mathbf{z})$, $\mathbf{y}=(y_\mathbf{z},Y_\mathbf{z})$ inside $\mathbf{z}^\perp$, we have $X_\mathbf{z}=Y_\mathbf{z}$, as $\mathbf{x}\approx^h\mathbf{y}$. Applying Lemma 4.3, we obtain a vertex $\mathbf{a}\in\{\mathbf{p},\mathbf{x},\mathbf{y}\}^\perp$. Writing $\mathbf{p}=(p_\mathbf{a},H_\mathbf{a})$ in $\mathbf{a}^\perp$, we see $p_\mathbf{a}\in X_\mathbf{a}$, whence $p_\mathbf{a}\in Y_\mathbf{a}$ by $\mathbf{x}\approx^h_\mathbf{a}\mathbf{y}$. But now we can find a vertex $\mathbf{p}_1=(p_\mathbf{a},H^1_\mathbf{a})$ with $y_\mathbf{a}\in H^1_\mathbf{a}$ and consequently $p\in\left\langle\mathbf{y}^\perp\right\rangle$.

Now assume that $\left\langle\mathbf{x}^\perp\right\rangle=\left\langle\mathbf{y}^\perp\right\rangle$ but $\mathbf{x}\not\approx^h\mathbf{y}$. Take a point $\mathbf{z}\in\{\mathbf{x},\mathbf{y}\}^\perp$. Fix points $x'\in Y_\mathbf{z}\backslash X_\mathbf{z}$ and $y'\in X_\mathbf{z}\backslash Y_\mathbf{z}$. Then $\mathbf{x}'=(x',X_\mathbf{z})\approx^h\mathbf{x}$ and $\mathbf{y}'=(y',Y_\mathbf{z})\approx^h\mathbf{y}$. So, by the first paragraph of this proof, we have $\left\langle\mathbf{x}'^\perp\right\rangle=\left\langle\mathbf{x}^\perp\right\rangle=\left\langle\mathbf{y}^\perp\right\rangle=\left\langle\mathbf{y}'^\perp\right\rangle$. However, $\mathbf{x}'\perp\mathbf{y}'$, so the global point containing $\mathbf{x}'$ is in $\langle\mathbf{x}'\rangle$ which contradicts Lemma 4.16. ∎

Lemma 4.18. *Let* $\mathbf{x}$ *be a vertex of* Γ*. Then* $\left\langle\mathbf{x}^\perp\right\rangle$ *is a hyperplane of* $\mathbb{P}_\Gamma$*.*

Proof. Suppose l is a global line of Γ. We have to show that it intersects $\left\langle\mathbf{x}^\perp\right\rangle$. Let $a\neq b$ be two global points on l and choose vertices $\mathbf{a}\in a$, $\mathbf{b}\in b$. By Lemma 3.12 we may assume $\mathbf{a}\perp\mathbf{b}$. By Proposition 4.5, there exists a vertex $\mathbf{y}$ with $\mathbf{b}\perp\mathbf{y}\perp\mathbf{x}$.

Changing $\mathbf{x}$ inside $\mathbf{y}^\perp$ while leaving $\left\langle\mathbf{x}^\perp\right\rangle$ invariant, we can assume $B_\mathbf{y}\cap X_\mathbf{y}\cap\langle b_\mathbf{y},x_\mathbf{y}\rangle=\emptyset$ (for $\mathbf{b}=(b_\mathbf{y},B_\mathbf{y})$, $\mathbf{x}=(x_\mathbf{y},X_\mathbf{y})$, inside $\mathbf{y}^\perp$); notice that, by Lemma 4.17, changing $\mathbf{x}$ as indicated means changing the point $x_\mathbf{y}$. Consequently, by Lemma 4.3, there exists a vertex $\mathbf{c}\in\{\mathbf{a},\mathbf{b},\mathbf{x}\}^\perp$. Now local analysis of $\mathbf{c}^\perp$ shows that l has to intersect $\left\langle\mathbf{x}^\perp\right\rangle$. Lemma 4.16 shows that $\left\langle\mathbf{x}^\perp\right\rangle$ is not the whole space, and $\left\langle\mathbf{x}^\perp\right\rangle$ is a hyperplane. ∎

By $\mathbb{H}_\Gamma$ we denote the set of all subsets $\left\langle\mathbf{x}^\perp\right\rangle$, where $\mathbf{x}$ runs through the vertex set of Γ.

Lemma 4.19. *The set* $\mathbb{H}_\Gamma$ *is a subspace of* ${\mathbb{P}_\Gamma}^{\mathrm{dual}}$ *such that* $\mathbb{H}_\Gamma$ *has trivial annihilator in* $\mathbb{P}_\Gamma$*.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two points of Γ with $\left\langle\mathbf{x}^\perp\right\rangle\neq\left\langle\mathbf{y}^\perp\right\rangle$. Denote by x and y the global points and by X and Y the global hyperplanes containing $\mathbf{x}$ and $\mathbf{y}$, respectively. By Proposition 4.5 there exists a third vertex adjacent to $\mathbf{x}$

and $\mathbf{y}$. Then, by Lemma 3.12, there exist adjacent vertices $\mathbf{x}_1 \in X$ and $\mathbf{y}_1 \in Y$ with $\left\langle \mathbf{x}^\perp \right\rangle = \left\langle \mathbf{x}_1^\perp \right\rangle$ and $\left\langle \mathbf{y}^\perp \right\rangle = \left\langle \mathbf{y}_1^\perp \right\rangle$. We will show that the hyperline on $\left\langle \mathbf{x}^\perp \right\rangle$ and $\left\langle \mathbf{y}^\perp \right\rangle$ is contained in $\mathbb{H}_\Gamma$. By the above we can assume that $\mathbf{x}$ and $\mathbf{y}$ are adjacent.

We will show that for every global point u, there is a point $\mathbf{z}$ such that $\left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle \subseteq \left\langle \mathbf{z}^\perp \right\rangle$ and $u \in \left\langle \mathbf{z}^\perp \right\rangle$. This proves $\mathbb{H}_\Gamma$ to be a subspace of ${\mathbb{P}_\Gamma}^{\mathrm{dual}}$.

Let Π be the hyperplane of $\mathbb{P}_\Gamma$ containing $\left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle$ and u. The global line on x and y meets Π in a point outside $\left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle$. So, without loss we may assume this intersection point to be u.

Let $\mathbf{w}$ be adjacent to both $\mathbf{x}$ and $\mathbf{y}$. Then both x and y are global points in $\left\langle \mathbf{w}^\perp \right\rangle$ and hence, as $\left\langle \mathbf{w}^\perp \right\rangle$ is a subspace, so is u. Inside $\mathbf{w}^\perp \cong \mathbf{H}(\mathbb{P}, \mathbb{H})$ we find a point $\mathbf{z}$ such that $\mathbf{z}^\perp$ meets all local points at $\mathbf{w}^\perp$ which meet $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{w}^\perp$ as well as u.

Suppose $p \in \left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle \cap \left\langle \mathbf{w}^\perp \right\rangle$. Let $\mathbf{p} = (p_\mathbf{w}, P_\mathbf{w}) \in p$ be a point in $\mathbf{w}^\perp$. Inside $\mathbf{w}^\perp$ we set $\mathbf{x} = (x_\mathbf{w}, X_\mathbf{w})$ and $\mathbf{y} = (y_\mathbf{w}, Y_\mathbf{w})$. As $p \in \langle \mathbf{x}^\perp \rangle$, there is a point $\mathbf{p}' \in p \cap \mathbf{x}^\perp$. Let $\mathbf{u}$ be adjacent to both $\mathbf{x}$ and $\mathbf{p}'$. Inside $\mathbf{u}^\perp$ we see that the local hyperplane containing $\mathbf{x}$ is incident to $p_\mathbf{u}$. Now Lemma 4.11 implies that $p_\mathbf{w}$ is incident with $X_\mathbf{w}$. Similarly, $p_\mathbf{w}$ is incident with $Y_\mathbf{w}$. But that implies that we can find an element in p which is in $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{w}^\perp$. In particular, we have found that $p \in \langle \mathbf{z}^\perp \rangle$. Thus $\left\langle \mathbf{z}^\perp \right\rangle$ contains $\left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle \cap \left\langle \mathbf{w}^\perp \right\rangle$ and w, the global point on $\mathbf{w}$, and hence $\left\langle \mathbf{x}^\perp \right\rangle \cap \left\langle \mathbf{y}^\perp \right\rangle$. Moreover, as $\mathbf{z}^\perp$ meets u, the hyperplane $\langle \mathbf{z}^\perp \rangle$ contains Π and hence coincides with Π.

It remains to show that the intersection of all elements in $\mathbb{H}_\Gamma$ is empty. However, that easily follows from Lemma 4.16. ∎

Lemma 4.20. *Suppose x is a global point in $\mathbb{P}_\Gamma$ and $H \in \mathbb{H}_\Gamma$ is a hyperplane not containing x. Then there is a vertex $\mathbf{x} \in x$ with $\left\langle \mathbf{x}^\perp \right\rangle = H$.*

Proof. Suppose $\mathbf{x} \in x$ and $\mathbf{y}$ is a vertex of Γ with $\left\langle \mathbf{y}^\perp \right\rangle = H$. Then in $\mathbf{z}^\perp$, for some common neighbor $\mathbf{z}$ of $\mathbf{x}$ and $\mathbf{y}$, we find a vertex $\mathbf{x}' \in x \cap Y$, where Y is the global hyperplane on $\mathbf{y}$. But then, by Lemma 4.17, $\left\langle \mathbf{x}'^\perp \right\rangle = H$. ∎

Proposition 4.21. *Let Γ be a connected, locally $\mathbf{H}(\mathbb{P}, \mathbb{H})$ graph. Then Γ is isomorphic to $\mathbf{H}(\mathbb{P}_\Gamma, \mathbb{H}_\Gamma)$.*

Proof. Consider the map $\Gamma \to \mathbf{H}(\mathbb{P}_\Gamma, \mathbb{H}_\Gamma) : \mathbf{x} \mapsto \left(x, \left\langle \mathbf{x}^\perp \right\rangle\right)$ where x is the global point of Γ containing $\mathbf{x}$. We want to show that this is an isomorphism of graphs. The map is well-defined by Lemma 4.18. Surjectivity follows from Lemma 4.20, since any point x of $\mathbb{P}_\Gamma$ is a global point of Γ and any hyperplane in $\mathbb{H}_\Gamma$ not containing it is of the form $\left\langle \mathbf{x}^\perp \right\rangle$ for a vertex $\mathbf{x} \in x$. Injectivity is obtained as follows. Suppose the global point x contains two vertices $\mathbf{x}_1$, $\mathbf{x}_2$ with $\left\langle \mathbf{x}_1^\perp \right\rangle = \left\langle \mathbf{x}_2^\perp \right\rangle$. By Proposition 4.5 there exists a vertex $\mathbf{y}$ adjacent to both $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $\left\langle \mathbf{x}_1^\perp \right\rangle = \left\langle \mathbf{x}_2^\perp \right\rangle$, both vertices describe the same hyperplane in $\mathbf{y}^\perp$. But they also describe the same point and hence have to be equal.

If $\mathbf{x} \perp \mathbf{y}$, then, letting x and y be the global points of Γ containing $\mathbf{x}$ and $\mathbf{y}$, respectively, we find $x \in \left\langle \mathbf{y}^\perp \right\rangle$ and $y \in \left\langle \mathbf{x}^\perp \right\rangle$, so $(x, \langle \mathbf{x}^\perp \rangle) \perp (y, \langle \mathbf{y}^\perp \rangle)$.

Finally assume that (x, X) and (y, Y) are adjacent points in $\mathbf{H}(\mathbb{P}_\Gamma, \mathbb{H}_\Gamma)$ with preimages $\mathbf{x}$ and $\mathbf{y}$, respectively. Let $\mathbf{z}$ be adjacent to both $\mathbf{x}$ and $\mathbf{y}$. Then inside $\mathbf{z}^\perp$ we find $\mathbf{x} = (x_\mathbf{z}, X_\mathbf{z})$ and $\mathbf{y} = (y_\mathbf{z}, Y_\mathbf{z})$. As x is incident with Y, and y with X, Lemma 4.11 implies that $x_\mathbf{z}$ is incident with $Y_\mathbf{z}$ and $y_\mathbf{z}$ with $X_\mathbf{z}$. This implies that $\mathbf{x}$ and $\mathbf{y}$ are adjacent. ∎

Theorem 1.1 is an immediate consequence of the above proposition and Lemma 4.19.

5. Small dimensions

Any connected, locally $\mathbf{H}_1$ graph admits an infinite universal cover and we obtain infinitely many counterexamples to local recognition of $\mathbf{H}_2$. The case of a locally $\mathbf{H}_2$ graph proves to be a bit more complicated. Here we provide a counterexample for $\mathbb{F} = \mathbb{F}_2$. The proof of its existence is based on a computation with the computer algebra package GAP [6].

Proposition 5.1. *There exists a connected graph on* $128 \cdot 120$ *vertices that is locally* $\mathbf{H}_2(2)$.

Proof. We determine the stabilizers of a vertex, an edge, and a 3-clique of the graph $\mathbf{H}_3(2)$ inside the canonical group $(\mathrm{P})\mathrm{SL}_4(2)$ and let GAP determine the order of the universal completion of the amalgam of these groups and their intersections. This universal completion is the group G with a presentation by the generators w, u, b, a and the relations

$$\begin{aligned}
&w^2 = u^2 = b^2 = a^2 = 1,\\
&(wu)^3 = (ab)^3 = 1,\\
&(bw)^3 = (bu)^4 = 1,\\
&(wub)^7 = (wa)^2 = (ua)^2 = 1.
\end{aligned}$$

The stabilizers of a vertex, an edge, and a 3-clique of $\mathbf{H}_3(2)$, respectively, are of the form

$$\begin{aligned}\langle w,u,b\rangle &\cong \mathrm{SL}_3(2),\\ \langle w,u,a\rangle &\cong \mathrm{SL}_2(2)\times 2,\\ \langle a,b\rangle &\cong \mathrm{Sym}_3,\end{aligned}$$

with the intersections

$$\begin{aligned}\langle w,u,b\rangle\cap\langle w,u,a\rangle &= \langle w,u\rangle \cong \mathrm{SL}_2(2),\\ \langle w,u,a\rangle\cap\langle a,b\rangle &= \langle a\rangle \cong 2,\\ \langle a,b\rangle\cap\langle w,u,b\rangle &= \langle b\rangle \cong 2.\end{aligned}$$

A coset enumeration in GAP shows that the order of G is $128\cdot|\mathrm{SL}_4(2)|$, and that there exists a normal subgroup $N\cong 2^{1+6}$ of G. Hence $\mathbf{H}_3(2)$ admits a 128-fold cover Γ with the same local structure. ∎

For each $m\in\mathbb{N}$, Erik Postma and the first author have found a construction of a graph which is a $(2^m)^6$-cover of the graph $\mathbb{H}_3(2^m)$, but which is locally isomorphic to $\mathbb{H}_2(2^m)$, see [3]. For $m=1$, this graph is a quotient of the graph described in the proposition. These results show that the bound on n in Theorem 1.1 is sharp.

6. Group-theoretic consequences

In this section we study group-theoretic consequences of our local recognition Theorem 1.1 of the point-hyperplane graphs $\mathbf{H}_n(\mathbb{F})$, where $n\geq 3$ is a finite integer and $\mathbb{F}$ a skew field. In particular, we prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Let G be a group as in the hypothesis of Theorem 1.2. We need to show that, up to isomorphism, $\mathrm{PSL}_{n+2}(\mathbb{F})\leq G/Z(G)\leq \mathrm{PGL}_{n+2}(\mathbb{F})$.

We use the notation of Theorem 1.2. By (iii) of Theorem 1.2, we can choose an element $z\in J\cap K$ that is a reflection in the groups J and K conjugate to x and y, respectively. Hence x is a reflection in J and y is a reflection in K. Note that z commutes with x and y. As, by (i), $K\cong\mathrm{GL}_{n+1}(\mathbb{F})$, we find the elements y and z to be conjugate in K by an involution. Similarly, by (ii), x and z are conjugate in J by an involution. Therefore the conjugation action of the group G induces an action as the group Sym_3 on the set $\{x,y,z\}$ and as the group Sym_2 on the set $\{x,y\}$. Consider the graph Γ on all conjugates of x in G. A pair a, b of vertices of Γ is adjacent if there exists an element $g \in G$ such that $(gxg^{-1}, gyg^{-1}) = (a,b)$. Since G induces the action of Sym_3 on $\{x,y,z\}$, this definition of adjacency is completely symmetric, and we have defined an undirected graph. The elements x, y, z form a 3-clique of Γ. Define U_1 to be the stabilizer in G of the vertex x, and define U_2 to be the stabilizer in G of the edge $\{x,y\}$. The stabilizer of $\{x,y\}$ permutes x and y and therefore interchanges $C_G(x) \geq K$ and $C_G(y) \geq J$, see (i) and (ii). Hence the stabilizer of x together with the stabilizer of $\{x,y\}$ generates G, as $G = \langle J,K\rangle \leq \langle U_1,U_2\rangle$. Consequently, the graph Γ is connected. Also, Γ is locally $\mathbf{H}_n(\mathbb{F})$ by construction. To prove this, it is enough to show that any triangle in Γ is a conjugate of (x,y,z). Let (a,b,c) be a triangle. Let $g \in G$ with $(gxg^{-1}, gyg^{-1}) = (a,b)$. Notice that $b, d = gzg^{-1} \in gKg^{-1}$ are commuting reflections of gKg^{-1}. The edges (a,b) and (a,c) are conjugate in $C_G(a) = gXg^{-1} \times gKg^{-1}$ (use (i) of Theorem 1.2). Choose $h \in C_G(a)$ such that $(hah^{-1}, hbh^{-1}) = (a,c)$. Then $h = h_X h_K$ with $h_X \in gXg^{-1}$, $h_K \in gKg^{-1}$. The element h_X centralizes b and d, since $b,d \in gKg^{-1}$. Therefore $c = hbh^{-1} = h_K b h_K^{-1} \in gKg^{-1}$ is a reflection of gKg^{-1}. Hence (a,b,d) and (a,b,c) are conjugate in $gKg^{-1} \cong \mathrm{GL}_{n+1}(\mathbb{F})$. Therefore (a,b,c) and (x,y,z) are conjugate in G.

Thus, by Theorem 1.1, the graph Γ is isomorphic to $\mathbf{H}_{n+1}(\mathbb{F})$. Moreover, there is a kernel N of the action of G on Γ, such that G/N can be embedded in $\mathrm{Aut}(\Gamma)$, which has been determined in Corollary 3.10. Since G/N is transitive on Γ and the stabilizer in G/N of the vertex x induces $\mathrm{PGL}_{n+1}(\mathbb{F})$ on the neighbors of x, we find that $\mathrm{PSL}_{n+2}(\mathbb{F}) \leq G/N$. Furthermore, as G is generated by $C_G(x)$ and $C_G(y)$, we find that G/N embeds in $\mathrm{PGL}_{n+2}(\mathbb{F})$.

Let $g \in N$. Then g acts trivially on Γ, in particular it centralizes x and y, so we have $g \in X \times K$ and $g \in Y \times J$. Let $g_X \in X$ and $g_K \in K$ be such that $g = g_X g_K$. The element g_X commutes with K, and therefore also centralizes all neighbors of x. Consequently, also $g_K = g_X^{-1} g$ centralizes all neighbors of x, and hence lies in the center of K. We have proved that g commutes with K. Similarly, g commutes with J. This implies that g commutes with $G = \langle J,K\rangle$, and, thus, $g \in Z(G)$. Certainly, $Z(G)$ acts trivially on Γ, whence $N = Z(G)$. ∎

The above proves Theorem 1.2. It only remains to prove Theorem 1.3. This will be done next. Its proof proceeds along the lines of the proof of Theorem 1.2.

Proof of Theorem 1.3. Let G be a group as in the hypothesis of Theorem 1.3, and retain the notation as in the hypotheses. We need to show that $G/Z(G) \cong \mathrm{PSL}_{n+2}(\mathbb{F})$.

The element z is conjugate to both x and y, so, also x and y are conjugate. Moreover, x and y are generalized reflections in J and K, respectively. Note that z commutes with x and y. As $K \cong \mathrm{SL}_{n+1}(\mathbb{F})$, we find the elements y and z to be conjugate in K by an involution. Similarly, x and z are conjugate in J by an involution. Therefore the conjugation action of the group G induces an action as the group Sym_3 on the set $\{x,y,z\}$ and as the group Sym_2 on the set $\{x,y\}$. Consider the graph Γ on all conjugates of $\langle x\rangle$ in G. A pair a, b of vertices of Γ is adjacent if there exists an element $g \in G$ such that $(g\langle x\rangle g^{-1}, g\langle y\rangle g^{-1}) = (a,b)$. As in the proof of Theorem 1.2, the graph Γ is connected.

Let (a,b,c) be a triangle of Γ. We will show that (a,b,c) is also conjugate to $(\langle x\rangle, \langle y\rangle, \langle z\rangle)$. Without loss of generality, we can assume that $a = \langle x\rangle$ and $b = \langle y\rangle$. The edges (a,b) and (a,c) are conjugate in $N_G(a)$. Choose $h \in N_G(a)$ such that $(hah^{-1}, hbh^{-1}) = (a,c)$. Since $C_G(a)$ is normal in $N_G(a)$, and K is characteristic in $C_G(x)$, we find that h normalizes K. Therefore $c = hbh^{-1}$ is a group of order p generated by a generalized reflection of K. As each generalized reflection torus is isomorphic to $\mathbb{F}^*/\langle \lambda \in \mathbb{F}^* \mid \lambda^{n+1} = 1\rangle$, it contains a unique subgroup of order p. But then $(b, \langle z\rangle)$ and (b,c) are conjugate inside $K \cong \mathrm{SL}_{n+1}(\mathbb{F})$. As $K \le C_G(a)$ we find the triangles (a,b,c) and $(\langle x\rangle, \langle y\rangle, \langle z\rangle)$ to be conjugate in G.

As each generalized reflection torus contains a unique subgroup of order p, we find Γ to be locally $\mathbf{H}_n(\mathbb{F})$. But that implies, by Theorem 1.1, that the graph Γ is isomorphic to $\mathbf{H}_{n+1}(\mathbb{F})$.

Let N be the kernel of the action of G on Γ. Then, as in the proof of Theorem 1.2, we see that $G/N \le \mathrm{PGL}_{n+2}(\mathbb{F})$. In particular, $K \cap N = 1$ and, since G is generated by J and K, we even have $G/N = \mathrm{PSL}_{n+2}(\mathbb{F})$. Moreover, as $N \le N_G(\langle x\rangle)$ and K is normal in $N_G(\langle x\rangle)$, we find $[N,K] \le K \cap N = 1$. Similarly, $[N,J] = 1$ and hence $N \le Z(\langle K,J\rangle) = Z(G)$, which completes the proof of the proposition, as $Z(G)$ is in the kernel of the action by construction of Γ. ∎

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